The 4^{th} "STARS of MATHEMATICS" Competition
December 11, 2010 ★★★ ICHB - Bucharest

Solutions

Problem 1. Let $D := \{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\}$. Prove there exists a set $S \subset D$ with $|S| \geq \left\lfloor \frac{3}{5} n(n+1) \right\rfloor$, such that for any $(x_1, y_1),\ (x_2, y_2) \in S$ we have $(x_1 + x_2, y_1 + y_2) \notin S$.

PETER J. CAMERON

Solution. It is easy to find a weaker bound of $|S| = n \left\lceil \frac{1}{2} n \right\rceil$ by taking $S = \{(x, y) \in D \mid x > n/2\}$. To find the bound asked, we need look at the *diagonals* of the tableau!

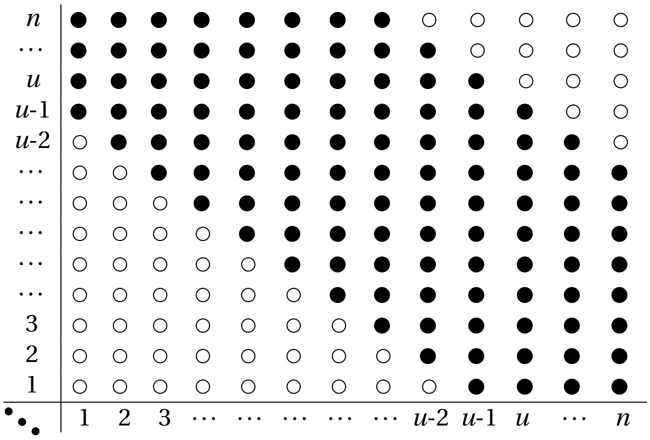


Diagram for the selection ● of S_u (exact for $n = 13$).

Define the diagonal $\Delta_k = \{(x, y) \in \mathbb{N} \times \mathbb{N} \mid x + y = k\}$, for all $k \geq 0$. It is clear that if $(x_1, y_1) \in \Delta_{k_1}$ and $(x_2, y_2) \in \Delta_{k_2}$ then $(x_1 + x_2, y_1 + y_2) \in \Delta_{k_1 + k_2}$.

Therefore $S_u := \bigcup_{u \leq k \leq 2u-1} (\Delta_k \cap D)$, for $u < n + 1 \leq 2u - 1$, has the defining property. Since we have $|\Delta_k \cap D| = k - 1$ for $2 \leq k \leq n + 1$, respectively $|\Delta_k \cap D| = 2n - (k - 1)$ for $n + 1 \leq k \leq 2n$, it follows that $|S_u| = \sum_{k=u}^{n} (k - 1) + \sum_{k=n+1}^{2u-1} (2n - (k - 1)) = -\frac{1}{2}(5u^2 - (8n+9)u + 2(n+1)(n+2))$. Thus the **maximal** value for $|S_u|$ is obtained at the nearest integer v to $\frac{8n+9}{10}$, thus at $v = \left\lfloor \frac{4n+7}{5} \right\rfloor$, for which $|S_v| = \left\lfloor \frac{3}{5} n(n+1) \right\rfloor$. ■

Remarks. The set D is a product poset of chains, also seen as being a *graded* poset – with elements appearing in its *Hasse diagram* on *levels* by their *rank*; the rank being precisely the sum of the coordinates (as some justification).[1]

Problem 2. Let $ABCD$ be a square, and points $M \in [BC]$, $N \in [CD]$, $P \in [DA]$, such that

$$\angle(\overrightarrow{AB}, \overrightarrow{AM}) = x,\ \ \angle(\overrightarrow{BC}, \overrightarrow{MN}) = 2x,\ \ \angle(\overrightarrow{CD}, \overrightarrow{NP}) = 3x.$$

i) Show that, for any $x \in [0, \pi/8]$, such a configuration uniquely exists, and P ranges over the entire segment $[DA]$;

ii) Determine the number of angles $x \in [0, \pi/8]$ for which

$$\angle(\overrightarrow{DA}, \overrightarrow{PB}) = 4x.$$

DAN SCHWARZ

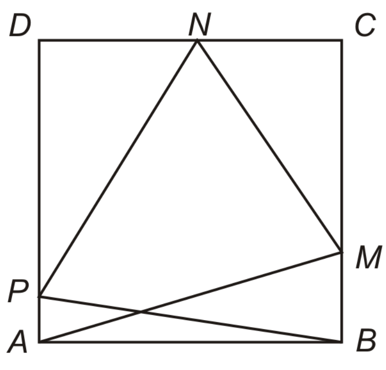
Solution. i) Assume $AB = 1$, and denote $t = \tan x$. Since

$$1 = \tan\frac{\pi}{4} = \frac{2\tan\frac{\pi}{8}}{1 - \tan^2\frac{\pi}{8}}$$

it follows $\tan\frac{\pi}{8}$ is the positive root of $t^2 + 2t - 1 = 0$, and so $\tan\frac{\pi}{8} = \sqrt{2} - 1$.

Now $BM = t$, hence M takes **once and only once all** values of the interval $[0, \sqrt{2} - 1]$, while $CN = (1 - t)\tan 2x = \frac{2t}{1+t}$, hence N takes all values of the interval $[0, 2 - \sqrt{2}]$. Lastly, we also get $DP = \left(1 - \frac{2t}{1+t}\right)\tan 3x = \frac{1-t}{1+t} \cdot \frac{t(3-t^2)}{1-3t^2} \geq 0$, whence $PA = 1 - \frac{1-t}{1+t} \cdot \frac{t(3-t^2)}{1-3t^2} = \frac{(t^2+2t-1)(t^2+1)}{(t+1)(3t^2-1)} \geq 0$, hence P takes **all** values of the full interval $[0, 1]$.

Therefore such a configuration will uniquely exist for any $x \in [0, \pi/8]$. In fact the points M, N and P, all three, vary monotonically.[2]



The position that must occur.[3]

ii) Take now $Q \in BC$ such that $\angle(\overrightarrow{DA}, \overrightarrow{PQ}) = 4x > 0$. Then

$$\begin{aligned} BQ &= (1 - PA\tan 4x)/\tan 4x \\ &= \left(1 - \frac{(t^2+2t-1)(t^2+1)}{(t+1)(3t^2-1)} \cdot \frac{4t(1-t^2)}{1-6t^2+t^4}\right) \frac{1-6t^2+t^4}{4t(1-t^2)} \\ &= \left(1 - \frac{4t(1-t)(t^2+1)}{(3t^2-1)(t^2-2t-1)}\right) \frac{(t^2+2t-1)(t^2-2t-1)}{4t(1-t^2)} \\ &= \frac{t^2+2t-1}{4t(1-t^2)(3t^2-1)}(7t^4-10t^3-2t+1). \end{aligned}$$

What we look for is $Q \equiv B$, i.e. $BQ = 0$ (as a matter of fact occurring for $x = \pi/8$, and seemingly no more realized). But $7t^4 - 10t^3 - 2t + 1 = (t^2+2t-1)(7t^2-24t+55) - 136t + 56 = (1-x)(2-7x^3) - (3x^3+1)$, thus decreasing on $[0, \sqrt{2}-1]$, while $56 < 136(\sqrt{2}-1)$, hence the polynomial considered above has **just one** root $0 < \tau \approx 0.3447 < 0.4142 \approx \sqrt{2}-1$, corresponding to **just one** angle $0 < \xi \approx 0.3319 < 0.3927 \approx \pi/8$.[4]

Thus the **two solutions** $x \in [0, \pi/8]$ are $\boxed{\xi \text{ and } \pi/8}$. ■

Problem 3. Find the largest constant $K \geq 0$ such that for any $0 \leq k \leq K$, and for any non-negative real numbers a, b, c, satisfying $a^2 + b^2 + c^2 + kabc = k + 3$, to have $a + b + c \leq 3$.

DAN SCHWARZ

Solution. Let us work from first principles. Whenever (at least) one variable is zero, say c, it follows $a^2 + b^2 = k + 3$, hence $a + b + c \leq \sqrt{2(k+3)} \leq 3$ for $k \leq 3/2$, with equality holding for $a = b = 3/2$. We thus have a starting tentative limiting bound of $\boxed{K = 3/2}$.

Let also notice that $a = b = c = 1$ checks for any value of k, while providing a maximal admissible value for $a + b + c = 3$.

Assume $\sigma = a + b + c > 3$. Notice that then (for some $k > 0$) $k(1 - abc) = a^2 + b^2 + c^2 - 3 \geq \frac{1}{3}(a+b+c)^2 - 3 = \frac{1}{3}\sigma^2 - 3 > 0$, by Cauchy-Schwarz, and so $abc < 1$. Now is the time for the **key step**. Since $a + b + c > 3$, assuming some ordering of our variables, say $0 \leq a \leq b \leq c$, it will follow $c > 1$, and so we can compute

$$\begin{aligned} k &= \frac{a^2+b^2+c^2-3}{1-abc} \\ &= \frac{(a+b)^2+c^2-3-2ab}{1-abc} \\ &= \frac{(\sigma-c)^2+c^2-3-2ab}{1-abc} \\ &= \frac{c(2c^2-2\sigma c+\sigma^2-3)-2+2(1-abc)}{c(1-abc)} \\ &= \frac{2}{c} + \frac{2c^3-2\sigma c^2+(\sigma^2-3)c-2}{c(1-abc)}. \end{aligned}$$

Denote $f(c) = 2c^3 - 2\sigma c^2 + (\sigma^2-3)c - 2$. As $(x-y)(f(x)-f(y)) = (x-y)^2\left(\frac{3}{2}\left(x+y-\frac{2}{3}\sigma\right)^2 + \frac{1}{2}(x-y)^2 + \frac{1}{3}(\sigma^2-9)\right) \geq 0$, f is increasing; and as $f(1) = (\sigma+1)(\sigma-3) > 0$, it follows $f(c) > 0$. Therefore the minimal value for k is reached when $ab = 0$, for $a = 0$, whence $k \geq b^2 + c^2 - 3 \geq \frac{1}{2}(b+c)^2 - 3 = \frac{1}{2}\sigma^2 - 3 > \frac{3}{2}$.

The issue of the extremal points, leading to equality when $k = 3/2$, will better be addressed within the next alternative solution. ■

Alternative Solution. Let us apply the trusted method of Lagrange multipliers. Define

$$L(a,b,c) = a + b + c - \lambda(a^2 + b^2 + c^2 + kabc - k - 3).$$

The analysis of the values on the border of the domain of L has been done above, leading to $K \leq 3/2$ (we could now simplify our work to just $k = 3/2$, but it is enlightening to see it in full generality). The system of partial derivatives is

$$\begin{cases} \dfrac{\partial L}{\partial a} = 1 - \lambda(2a + kbc) \\ \dfrac{\partial L}{\partial b} = 1 - \lambda(2b + kca) \\ \dfrac{\partial L}{\partial c} = 1 - \lambda(2c + kab) \end{cases}$$

Equaling the partial derivatives to zero forbids $\lambda = 0$. Then, from pairwise equalities, we get

$$\lambda(a-b)(2-kc) = \lambda(b-c)(2-ka) = \lambda(c-a)(2-kb) = 0.$$

One possibility is $a = b = c = x$, thus (from the constraint) $3x^2 + kx^3 = k + 3$, or $(x-1)(kx^2 + (k+3)x + (k+3)) = 0$, with only non-negative real solution $x = 1$, since the coefficients of the quadratic factor are non-negative. Then $a + b + c = 3x = 3$, the admissible maximum.

The other possibility is for two variables to be equal, say $a = b$, but not equal to the third, hence needing $a = b = 2/k$, thus (from the constraint) $8/k^2 + c^2 + 4c/k = k + 3$, or $k^2c^2 + 4kc - (k^3 + 3k^2 - 8) = 0$, with non-negative real solution $c = \frac{(k+2)\sqrt{k-1}-2}{k}$ for $k^3 + 3k^2 - 8 \geq 0$, i.e. $k \geq \kappa \approx 1.3553$. Then $a + b + c = \frac{(k+2)\sqrt{k-1}+2}{k} < 3$ for $k < 2$, since it is equivalent to $(k-2)^3 < 0$. As k needs be at most $K \leq 3/2$, these points are critical, but not global maxima (in fact they turn to be global minima).[5]

Putting it all together, the largest admissible value for K turns to be $3/2$, with the maximum value $a + b + c = 3$ being reached only at points $(3/2, 3/2, 0), (3/2, 0, 3/2), (0, 3/2, 3/2)$ and $(1,1,1)$. For $0 \leq k < 3/2$ the unique maximum is reached at $(1,1,1)$. Notice the importance of examining the values on the border; without that, the other critical interior points found but $(1,1,1)$ (which works for any k) achieve a larger value than 3 for $a + b + c$ only starting with $k > 2$, so would induce the erroneous bound $K = 2$. ■

Alternative Solution. The fact the value 3 for $a + b + c$ is reached for $k = 3/2$ both at $a = b = c = 1$, and at $a = b = 3/2$ and $c = 0$ et al., suggests this is a Schur-type inequality. Indeed, assume $0 \leq k \leq 3/2$ and $\sum a > 3$.

Then

$$
\begin{aligned}
k+3 &= \sum a^2 + kabc \\
&\geq \sum a^2 + \frac{3kabc}{\sum a} \\
&= \left(1-\frac{k}{3}\right)\sum a^2 + \frac{k}{3}\left(\sum a^2 + \frac{9abc}{\sum a}\right) \\
&\geq \left(1-\frac{k}{3}\right)\sum a^2 + \frac{2k}{3}\sum ab \\
&= \left(1-\frac{2k}{3}\right)\sum a^2 + \frac{k}{3}\left(\sum a^2 + 2\sum ab\right) \\
&= \left(1-\frac{2k}{3}\right)\sum a^2 + \frac{k}{3}\left(\sum a\right)^2 \\
&\geq \left(\frac{1}{3}\left(1-\frac{2k}{3}\right)+\frac{k}{3}\right)\left(\sum a\right)^2 \\
&= \frac{k+3}{9}\left(\sum a\right)^2 \\
> k+3
\end{aligned}
$$

since inequality $\sum a^2 + \dfrac{9abc}{\sum a} \geq 2\sum ab$ is in turn equivalent to $\left(\sum a^2\right)\left(\sum a\right) + 9abc \geq 2\left(\sum ab\right)\left(\sum a\right)$, then $\sum a^3 + 3abc \geq \sum a^2b + \sum ab^2$, at last $\sum a(a-b)(a-c) \geq 0$, a basic form of Schur; while $\sum a^2 \geq \frac{1}{3}\left(\sum a\right)^2$ by Cauchy-Schwarz. ■

Problem 4. Let a, b, c be given positive integers. Prove there exists some positive integer N such that

$$
\begin{aligned}
a &\mid Nbc + b + c \\
b &\mid Nca + c + a \\
c &\mid Nab + a + b
\end{aligned}
$$

if and only if, denoting $d = \gcd(a, b, c)$ and $a = dx$, $b = dy$, $c = dz$, the positive integers x, y, z are pairwise co-prime, and also $\gcd(d, xyz) \mid x + y + z$.

DAN SCHWARZ

Solution. The necessity of having x, y, z be pairwise co-prime is proved by say, assuming $\gcd(x, y) > 1$.

Then $a \mid Nbc + b + c$ becomes $x \mid dNyz + y + z$, and so we must have $\gcd(x, y) \mid z$, absurd, since under this assumption it then follows $\gcd(x, y) \mid \gcd(x, y, z) = 1$.

On the other hand, if the co-primality condition holds, consider the integers

$$xyz - \sum x < 2xyz - \sum x < \cdots < \left(\sum xy\right)xyz - \sum x.$$

These $\sum xy$ integer numbers will yield different remainders modulo $\sum xy$, since if $ixyz - \sum x \equiv jxyz - \sum x \pmod{\sum xy}$, then also $\sum xy \mid |i-j|xyz$, whence $i = j$, since we have $0 \leq |i-j| < \sum xy$ and $\gcd(xyz, \sum xy) = 1$. Therefore there will exist some (unique) $1 \leq t \leq \sum xy$ such that $\sum xy \mid txyz - \sum x$, i.e. $txyz - \sum x = C\sum xy$ for some positive integer C, therefore $txyz = C\sum xy + \sum x$, so $x \mid Cyz + y + z$ et al. We found a suitable value C for the triplet x, y, z (for similar relations with the ones sought for a, b, c). Then all the other suitable values must be of the form $C' = C + \mathscr{M}xyz$, since we need have $xyz \mid (C' - C)\sum xy$, while $\gcd(xyz, \sum xy) = 1$.

Now the time has come to analyze the last condition. In order to have $a \mid Nbc + b + c$, and the similar others, equivalent to $x \mid dNyz + y + z$ et al., we need have $dN = C + Mxyz$ for some non-negative integer M. Denote $e = \gcd(d, xyz)$; then $e \mid d$, so $e \mid C + Mxyz$. But then we also must have $e \mid xyz \mid (C + Mxyz)\sum xy + \sum x$, hence $e \mid \sum x$.

Conversely, if $e \mid \sum x$, then $e \mid xyz \mid (C + Mxyz)\sum xy + \sum x$, so $e \mid C\sum xy$. Since clearly $\gcd(e, \sum xy) = 1$, this means $e \mid C$. Therefore we need have $\dfrac{d}{e}N = \dfrac{C}{e} + M\dfrac{xyz}{e}$, and since clearly $\gcd\left(\dfrac{d}{e}, \dfrac{xyz}{e}\right) = 1$, take $M \equiv -\dfrac{C}{e}\left(\dfrac{xyz}{e}\right)^{-1} \pmod{d/e}$, wherefore $\dfrac{d}{e}$ divides $\dfrac{C}{e} + M\dfrac{xyz}{e}$. Take now $N = \dfrac{C + Mxyz}{d}$ (of course, $N' = N + \mathscr{M}abc$ also works). ■

Remarks. Notice there exist easy counterexamples for $\gcd(x, y, z) = 1$, but x, y, z not pairwise co-prime; just take $x = pq$, $y = qr$, $z = rp$, with p, q, r prime.

Notice also there exist counterexamples to contradict the last condition, for example $(a, b, c) = (6, 9, 15)$, when $d = 3$, $e = 3$, $\sum x = 10$, and so $e \nmid \sum x$.

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[1] Actually it is a rank-unimodal and rank-symmetric poset; a seminal result is de Bruijn-Tengbergen-Kruyswijk theorem. The common algebraic-combinatorial jargon is the set S is *sum-free*, or $(S+S) \cap S = \emptyset$ (Minkowski sumset notation).

[2] An alternative proof uses the fact that such a configuration, with $P \equiv A$, can occur if (and only if) $x = \pi/8$ (hence P cannot bypass A for angles $0 \leq x \leq \pi/8$). Limit cases are $P \equiv D$ for $x = 0$ and $P \equiv A$ for $x = \pi/8$, so by continuity P ranges over the entire segment $[DA]$. The monotony of the variation of M, N and P must be argued by stronger methods than just continuity.

[3] Drawing courtesy of M. Bălună. It depicts the other solution ξ than $\pi/8$, which is shown to exist via trigonometry.

[4] For angle $x \in (\xi, \pi/8)$ the line PQ does not meet anymore the segment (AB), going beyond B, since Q **does not** vary monotonically. This phenomenon, I humbly reckon, can only be fathomed through analytic methods.

[5] For the record, for the values $k^3 + 3k^2 - 8 \geq 0$ we also have $\dfrac{(k+2)\sqrt{k-1}+2}{k} \leq \sqrt{2(k+3)}$, since it turns to be equivalent to $\left(2\dfrac{k^3+3k^2-8}{k\sqrt{2(k+3)}+4}\right)^2 \geq 0$. Let $\kappa \approx 1.3553$ be the unique real root of $k^3+3k^2-8=0$, with $\kappa \in (1, 3/2)$. Then for $k \in (\kappa, 3/2)$ we have the points on the border being local maxima, the other being global minima, with point $(1,1,1)$ as unique global maximum.