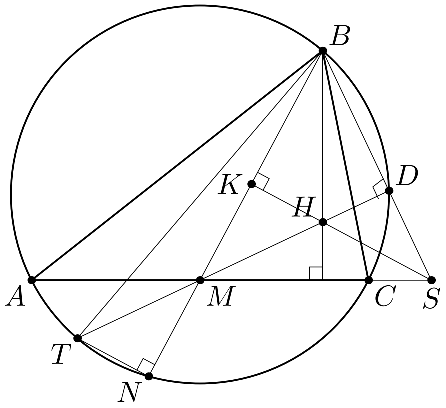
The 5^{th} "STARS of MATHEMATICS" Competition – Seniors
December 10, 2011 ★★★ ICHB – Bucharest

Solutions

Problem 1. Let ABC be an acute-angled triangle with $AB \neq BC$, M the midpoint of AC, N the point where the median BM meets again the circumcircle of $\triangle ABC$, H the orthocentre of $\triangle ABC$, D the point on the circumcircle for which $\angle BDH = 90°$, and K the point that makes $ANCK$ a parallelogram. Prove the lines AC, KH, BD are concurrent.

MOFM 2011 SHORTLIST - I. NAGEL



Solution. (Ilya Bogdanov) Let T be the diametrically opposite point to B on the circumcircle of $\triangle ABC$. Then $AT \perp AB$, $AT \perp CH$ and $CT \perp CB$, $CT \perp AH$, hence $ATCH$ is a parallelogram, and therefore M is the midpoint of HT. Since $DH \perp BD$, the line DH also passes through T; in other words, points M, T, H and D are collinear. Moreover, the segments TN and HK are symmetrical at M, and $TN \perp BN$; hence also $HK \perp BN$. Finally, denote by S the meeting point of KH and AC. Therefore BH and SH are the altitudes of the triangle BMS. Then MH is also its altitude, $MH \perp BS$, thus D lies on BS. ■

Alternative Solution. (Ionuţ Onişor) It is well-known the symmetrical of the orthocentre H of $\triangle ABC$ with respect to the midpoint M of AC is the diametrically opposite point to B on the circumcircle of $\triangle ABC$; let us denote this point by T. Thus the line DH passes through the points M and T, so clearly K is the symmetrical of N with respect to M, and lies on the median line BM. We get $\angle HKM = \angle TNM = 90°$. Therefore, for S the meeting point of the lines AC and BD, it follows H is the orthocentre of $\triangle BMS$, and then the line KH must pass through S. ■

Problem 2. Prove there do exist infinitely many positive integers n such that if a prime p divides $n(n+1)$ then p^2 also divides it (all primes dividing $n(n+1)$ bear exponent at least two). Exhibit (at least) two values, one even and one odd, for such numbers $n > 8$.

PÁL ERDÖS & KURT MAHLER

Solution. (Dan Schwarz) Let's try to find infinitely many n such that $n(n+1) = 2m^2$, with m even. This we write as $8m^2+1 = 4n^2+4n+1 = (2n+1)^2$, hence we must look at the solutions of the Pell equation $(2n+1)^2 - 8m^2 = 1$ having m even. Its primitive solution is $(2n+1, m) = (x_1, y_1) = (3, 1)$.

If we denote $(3+\sqrt{8})^k = x_k + y_k\sqrt{8}$, we can easily reach the recurrence relation(s) $x_{k+1} = 3x_k + 8y_k$ and $y_{k+1} = x_k + 3y_k$. But $x_1 = 3$ is odd, hence x_k is odd for all $k \geq 1$. Thus y_{k+1} and y_k have different parity, and since $y_1 = 1$ is odd, it follows y_{2k} is even.

In fact, since $(3+\sqrt{8})^2 = 17+6\sqrt{8}$, denoting $(3+\sqrt{8})^{2k} = A_k + B_k\sqrt{8}$ we get in a similar way as above the system $A_{k+1} = 17A_k + 48B_k$, $B_{k+1} = 6A_k + 17B_k$, and we can take $n = \dfrac{A_k - 1}{2}$. The characteristic polynomial is $\lambda^2 - 34\lambda + 1 = 0$, whence the recurrence $A_{k+2} = 34A_{k+1} - A_k$. Since we can start with $A_0 = 1$, the first three meaningful values for A_k are $17, 2\cdot 17\cdot 17 - 1, 2\cdot 17(2\cdot 17\cdot 17 - 1) - 17$, to which correspond the values $\boxed{n = 8,\ n = 288,\ n = 9800}$ (always even).

A simpler, more direct approach, would be to start with the eligible pair $(n, n+1) = (8, 9)$, and build another eligible one $(4n(n+1), (2n+1)^2)$.

We could alternatively have started with a Pell equation $(2n+1)^2 - 12m^2 = 1$ having $3 \mid m$. Its primitive solution is $(2n+1, m) = (x_1, y_1) = (7, 2)$, and $(7+2\sqrt{12})^3 = 1351 + 390\sqrt{12}$ provides the solution $\boxed{n = 675}$. ■

Problem 3. For a given integer $n \geq 3$, determine the range of values for the expression

$$E_n(x_1, x_2, \ldots, x_n) := \frac{x_1}{x_2} + \frac{x_2}{x_3} + \cdots + \frac{x_{n-1}}{x_n} + \frac{x_n}{x_1}$$

over real numbers $x_1, x_2, \ldots, x_n \geq 1$ satisfying $|x_k - x_{k+1}| \leq 1$ for all $1 \leq k \leq n-1$. Do also determine when the extremal values are achieved.

MOFM 2011 SHORTLIST - DAN SCHWARZ

Solution. We claim that $\boxed{\max E_n = 2n - H_n} \approx 2n - \ln n$, reached when $x_k = k$, for $1 \leq k \leq n$; here $H_n = 1 + \dfrac{1}{2} + \cdots + \dfrac{1}{n}$ is the partial sum of the harmonic series. On the other hand, clearly $\boxed{\min E_n = n}$ by AM-GM, reached when $x_k = x \geq 1$, for $1 \leq k \leq n$. By continuity, all intermediate values are also taken. The claim is now proved by simple induction.

For $n = 2$, take $1 \leq x \leq y \leq x+1$. Then $\dfrac{x}{y} + \dfrac{y}{x} = \dfrac{x^2+y^2}{xy} = \dfrac{5xy - [y + (y-x)][(x-1) + (x+1-y)]}{2xy} \leq \dfrac{5}{2} = 2\cdot 2 - \left(1 + \dfrac{1}{2}\right)$, with equality for $x = 1$ and $y = 2$ (or, equivalently, for $x = 2$ and $y = 1$ if we start with $1 \leq y \leq x \leq y+1$).

To be able to pursue this line by induction, notice that $E_{n+1} = E_n - \dfrac{x_n}{x_1} + \dfrac{x_n}{x_{n+1}} + \dfrac{x_{n+1}}{x_1}$. Take, for the ease of notation, $x = x_1$, $y = x_n$, $z = x_{n+1}$, with $x, y, z \geq 1$ and $|y - z| \leq 1$. Then it is enough to check $\dfrac{z(z-y) + xy}{xz} \leq 2 - \dfrac{1}{n+1} = 1 + \dfrac{n}{n+1}$, i.e. $(z-x)(z-y) \leq \dfrac{n}{n+1}xz$. Distinguish now two cases.

• If $z \geq x$, then we only need check it for $0 \leq z - y \leq 1$. It is thus enough to have $z - x \leq \dfrac{n}{n+1} xz$, or $\dfrac{z-x}{xz} \leq \dfrac{n}{n+1}$. But $z - x = x_{n+1} - x_1 = \sum_{k=1}^{n} (x_{k+1} - x_k) \leq \sum_{k=1}^{n} |x_{k+1} - x_k| \leq n$. Therefore $\dfrac{z-x}{xz} = \dfrac{1}{x} - \dfrac{1}{z} \leq \dfrac{1}{x} - \dfrac{1}{n+x} = \dfrac{n}{x(n+x)} \leq \dfrac{n}{n+1}$, with equality when $x = 1$, $z = n+1$; moreover, when $x_k = k$ for all $1 \leq k \leq n+1$. This meshes well with the equality case for E_n, and so accounts for the only general equality case.

• If $z \leq x$, then we only need check it for $0 \leq y - z \leq 1$. It is thus enough to have $x - z \leq \dfrac{n}{n+1} xz$, or $\dfrac{x-z}{xz} \leq \dfrac{n}{n+1}$. But $x - z = x_1 - x_{n+1} = \sum_{k=1}^{n} (x_k - x_{k+1}) \leq \sum_{k=1}^{n} |x_k - x_{k+1}| \leq n$. Therefore $\dfrac{x-z}{xz} = \dfrac{1}{z} - \dfrac{1}{x} \leq \dfrac{1}{z} - \dfrac{1}{n+z} = \dfrac{n}{z(n+z)} \leq \dfrac{n}{n+1}$, with equality when $z = 1$, $x = n+1$; moreover, when $x_k = n+2-k$ for all $1 \leq k \leq n+1$. This does not mesh well with having the equality case for E_n (and so it is not accounted for). ■

Problem 4. Given n sets A_i, with $|A_i| = n$, prove they may be indexed $A_i = \{a_{i,j} \mid j = 1,2,\ldots,n\}$, in such way that the sets $B_j = \{a_{i,j} \mid i = 1,2,\ldots,n\}$, $1 \leq j \leq n$, also have $|B_j| = n$.[1]

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Solution. (Dan Schwarz) We shall prove a slightly more general case, when $|A_i| = m > 0$, $A_i = \{a_{i,j} \mid j = 1,2,\ldots,m\}$**, and no element belongs to more than** m **sets.** The base case $n = 1$ of induction by n is trivial. Now, for $n \geq 2$, use the induction hypothesis for the sets A_i, $1 \leq i \leq n-1$ in order to build a $(n-1) \times m$ matrix, and build the row R_n with the elements of A_n (in some order). This creates a $n \times m$ matrix where the only way a column may contain duplicate entries (*bad* column) is for its entry on row R_n to be duplicated on some other row.

Moreover, from all the $n \times m$ matrices built this way, let us select one with **minimal** number of bad columns. If none, we are done, so assume there is at least one bad column C, with entry a on row R_n duplicated on some other row. Then there must exist some other column C' with no entry equal to a (since a can appear at most m times in the matrix). Denote by $\overline{C}$ the first $n-1$ elements of the column C and similarly $\overline{C'}$, and notice their elements are distinct, by the induction hypothesis. The column C' must be good, otherwise we swap elements on row R_n between the columns C and C', and C' becomes good from bad, contradicting the minimality of the number of bad columns.

Let us define $\varphi \colon \overline{C} \to \overline{C'}$ by $\varphi(c_i) = c'_i$ (so each entry in $\overline{C}$ is mapped onto the corresponding entry in $\overline{C'}$ which is situated on the same row). Define an *alternating path* between the two columns as being a sequence of entries $c_1\varphi(c_1)c_2\varphi(c_2)\ldots c_k\varphi(c_k)$, with c_i entries in $\overline{C}$, and its *swap* to be the replacing of entries c_i with $\varphi(c_i)$ and vice-versa. By the above, the swap of the alternating path $a\varphi(a)$ leaves C' a good column. Now we must assume $\varphi(a)$ exists as an entry in $\overline{C}$, otherwise C becomes good. Consider now the swap of the alternating path $a\varphi(a)\varphi(a)\varphi^2(a)$; it also leaves the column C' good, so we must assume $\varphi^2(a)$ exists as an entry in $\overline{C}$, otherwise C becomes good.

Iterating this procedure obliges $\varphi^{k-1}(a)$ to be an entry in $\overline{C}$, otherwise we can operate the appropriate swap and turn C into a good column, while leaving the column C' good. But $\varphi^{n-1}(a)$ cannot be an entry in $\overline{C}$, since there is no more room left (a appears as an entry in $\overline{C}$), thus the procedure must stop for some $k \leq n-1$, and the corresponding swap turns C into a good column, while leaving the column C' good, thereby contradicting the minimality of the number of bad columns invoked at the start. ■

Alternative Solution. (Ilya Bogdanov) Consider the bipartite graph G whose left shore A of vertices is made of the sets A_i, $1 \leq i \leq n$, whose right shore B of vertices is made of the elements of $\bigcup_{i=1}^{n} A_i$, and whose edges are those $\{A_i, a\}$ such that $a \in A_i$. All vertices in A have degree equal to n, while all vertices in B have degree at most n, hence $\Delta(G) = n$. Define $\chi'(G)$ to be the *edge-chromatic number* (or *chromatic index*) of G, i.e. the least integer k for which the edges of G can be colored using k colors in such way that any two adjacent edges bear different colors.

Clearly, every graph G satisfies $\chi'(G) \geq \Delta(G)$. For bipartite graphs we have equality, by dint of

Theorem 5.3.1 (König 1916) [R. DIESTEL - *Graph Theory*] *Every bipartite graph G satisfies* $\chi'(G) = \Delta(G)$.

Therefore n colors c_j, $1 \leq j \leq n$, are enough. Now we can denote by $a_{i,j}$ the element of A_i connected to it by the edge of color c_j. ■

Remark. Let us prove that in fact the result obtained is as strong as König's theorem, and turn it into an

Alternative Proof of König's theorem. Let A be a shore of G containing a vertex of maximal degree $\Delta(G)$. Do now "saturate" the graph G, by adding, for any element a in A of degree less than maximal, new elements in the other shore B and edges between them and a, so that every vertex in A will get maximal degree $\Delta(G)$. Now the result above, for $n = |A|$ and $m = \Delta(G)$, yields a coloring with $\Delta(G)$ colors, represented by the columns of the $n \times m$ matrix. Do now remove back the new vertices added in B (together with the new edges incident at them); this leaves a coloring with $\Delta(G)$ colors for the graph G. □

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[1] This is tantamount to building a $n \times n$ matrix, with the entries of row R_i made by the (distinct) elements of A_i, such that the columns C_j are also each made by distinct entries. Under this wording, the requirement becomes even more appealing.

The result is also mentioned, without proof, within Chapter 27 of [M. AIGNER, G. ZIEGLER - *Proofs from the Book*].