Olimpiada de Matematică –etapa locală- Galați

11 februarie 2012

Clasa a XI-a

Barem de evaluare

- **Pentru orice soluție corectă, chiar dacă este diferită de cea din barem, se acordă punctajul maxim corespunzător.**
- **Nu se acordă fracțiuni de punct, dar se pot acorda punctaje intermediare pentru rezolvări parțiale, în limitele punctajului indicat în barem.**

Nr. Problemă	**Soluție, rezolvare**	**Punctaj**
1.	$n=1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n}+n^{\alpha}\cdot x_n \Leftrightarrow x_n=\frac{n-\left(1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n}\right)}{n^{\alpha}}$ $=\frac{\left(1-\frac{1}{2}\right)+\left(1-\frac{1}{3}\right)+...+\left(1-\frac{1}{n}\right)}{n^{\alpha}}=$ $=\frac{\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+...+\frac{n-1}{n}}{n^{\alpha}}.$	**2p**
	Pentru $\alpha>0,$ cu Stolz-Cesaro obtinem: $\lim\limits_{n\to\infty}x_n=\lim\limits_{n\to\infty}\frac{\frac{n}{n+1}}{(n+1)^{\alpha}-n^{\alpha}}\ (1)$ $\lim\limits_{n\to\infty}\left[(n+1)^{\alpha}-n^{\alpha}\right]=\lim\limits_{n\to\infty}n^{\alpha}\cdot\left[\left(1+\frac{1}{n}\right)^{\alpha}-1\right]=\lim\limits_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)^{\alpha}-1}{\frac{1}{n}}\cdot n^{\alpha-1}=\alpha\cdot\lim\limits_{n\to\infty}n^{\alpha-1}=$ $=\begin{cases}\infty,\ \alpha>1\\ 1,\ \alpha=1\\ 0,\ \alpha\in(0,1)\end{cases}$ Pentru $\alpha\in(0,1)\Rightarrow(x_n)$ este divergent; Pentru $\alpha\in[1,\infty)\Rightarrow(x_n)$ este convergent.	**3p**
	$\alpha=0\Rightarrow x_n=\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+...+\frac{n-1}{n}\geq(n-1)\cdot\frac{1}{2},\ (\forall)n\geq 2\Rightarrow(x_n)$ divergent	**1p**

<table>
<tr>
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<td>$\alpha < 0$, notam $-\alpha=\beta$, $\beta>0$. Atunci $n=1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}+\frac{x_n}{n^{\beta}} \Rightarrow n^{\beta+1}=$
$=n^{\beta}\cdot\left(1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}\right)+x_n \Rightarrow x_n=n^{\beta}\cdot\left(n-\left(1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}\right)\right)=$
$=n^{\beta}\cdot\left(1+\frac{2}{3}+\frac{3}{4}+\dots+\frac{n-1}{n}\right)>n^{\beta}\cdot\frac{n-1}{2}\to\infty\Rightarrow\left(x_n\right)$
este divergent pentru $\alpha<0$.
In concluzie $\left(x_n\right)$ este convergent $\Leftrightarrow \alpha\geq 1$
$\left(x_n\right)$ este divergent $\Leftrightarrow \alpha<1$.</td>
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<td rowspan="2">2.</td>
<td>$\lim_{n\to\infty}\left[n^2\cdot\left(\sqrt[n]{a}-\sqrt[n+1]{a}\right)\right]=\lim_{n\to\infty}\left[n^2\cdot\left(a^{\frac{1}{n}}-a^{\frac{1}{n+1}}\right)\right]=\lim_{n\to\infty}\left[n^2\cdot a^{\frac{1}{n+1}}\cdot\left(a^{\frac{1}{n\cdot(n+1)}}-1\right)\right]=$
$\lim_{n\to\infty}\left[n^2\cdot a^{\frac{1}{n+1}}\cdot\frac{a^{\frac{1}{n\cdot(n+1)}}-1}{\frac{1}{n\cdot(n+1)}}\cdot\frac{1}{n\cdot(n+1)}\right]=\ln a.$</td>
<td>4p</td>
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<td>$\sqrt[n]{a}-2\cdot\sqrt[n+1]{a}+\sqrt[n+2]{a}=a^{\frac{1}{n}}-a^{\frac{1}{n+1}}-a^{\frac{1}{n+1}}+a^{\frac{1}{n+2}}=a^{\frac{1}{n+1}}\cdot\left(a^{\frac{1}{(n+1)\cdot n}}-1\right)-a^{\frac{1}{n+2}}\left(a^{\frac{1}{(n+1)\cdot(n+2)}}-1\right)\cdot$
$=a^{\frac{1}{n+2}}\cdot\left[a^{\frac{1}{(n+1)\cdot(n+2)}}\left(a^{\frac{1}{(n+1)\cdot n}}-1\right)-\left(a^{\frac{1}{(n+1)\cdot(n+2)}}-1\right)\right]=$
$a^{\frac{1}{n+2}}\cdot\left[\left(a^{\frac{1}{(n+1)\cdot(n+2)}}-1\right)\cdot\left(a^{\frac{1}{(n+1)\cdot n}}-1\right)+a^{\frac{1}{(n+1)\cdot n}}-a^{\frac{1}{(n+1)\cdot(n+2)}}\right]=$
$=a^{\frac{1}{n+2}}\cdot\left[\left(a^{\frac{1}{(n+1)\cdot(n+2)}}-1\right)\cdot\left(a^{\frac{1}{(n+1)\cdot n}}-1\right)+a^{\frac{1}{(n+1)\cdot(n+2)}}\cdot\left(a^{\frac{2}{(n+1)\cdot(n+2)\cdot n}}-1\right)\right].$
Se stie ca $\lim_{n\to\infty}\sqrt[n]{a}=1$ si $\lim_{n\to\infty}\frac{a^{x_n}-1}{x_n}=\ln a,$ unde $x_n\to 0.$
$n^3\cdot\left(a^{\frac{1}{(n+1)\cdot(n+2)}}-1\right)\cdot\left(a^{\frac{1}{(n+1)\cdot n}}-1\right)=n^3\cdot\frac{a^{\frac{1}{(n+1)\cdot(n+2)}}-1}{\frac{1}{(n+1)\cdot(n+2)}}\cdot\frac{a^{\frac{1}{(n+1)\cdot n}}-1}{\frac{1}{(n+1)\cdot n}}\cdot\frac{1}{(n+1)\cdot(n+2)}\cdot\frac{1}{(n+1)\cdot n}\to 0$ si
$n^3\cdot\frac{a^{\frac{2}{(n+1)\cdot(n+2)\cdot n}}-1}{\frac{2}{(n+1)\cdot(n+2)\cdot n}}\cdot\frac{2}{(n+1)\cdot(n+2)\cdot n}\to 2\cdot\ln a.$
Limita este $2\cdot\ln a$.</td>
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3.	$$\Delta=\begin{vmatrix}\sin A & \sin B & \sin C\\ 2\cdot\sin A\cdot\cos A & 2\cdot\sin B\cdot\cos B & 2\cdot\sin C\cdot\cos C\\ \sin A\cdot\left(4\cos^2 A-1\right) & \sin B\cdot\left(4\cos^2 B-1\right) & \sin C\cdot\left(4\cos^2 C-1\right)\end{vmatrix}=$$ $$=2\cdot\sin A\cdot\sin B\cdot\sin C\cdot\begin{vmatrix}1 & 1 & 1\\ \cos A & \cos B & \cos C\\ 4\cos^2 A-1 & 4\cos^2 B-1 & 4\cos^2 C-1\end{vmatrix}=$$ $$=8\cdot\sin A\cdot\sin B\cdot\sin C\cdot\begin{vmatrix}1 & 1 & 1\\ \cos A & \cos B & \cos C\\ \cos^2 A & \cos^2 B & \cos^2 C\end{vmatrix}=$$ $$8\cdot\sin A\cdot\sin B\cdot\sin C\cdot\left(\cos A-\cos B\right)\cdot\left(\cos B-\cos C\right)\cdot\left(\cos C-\cos A\right).$$	**5p**
	$\Delta=0\Leftrightarrow A=B$ sau A=C sau B=C $\Leftrightarrow$ ABC isoscel.	**2p**
4.	**a).** $P\left(X\right)=\det\left(A+X\cdot B\right)=\det A+\alpha\cdot X+\det B\cdot X^2,\ \alpha\in\mathbb{R}.$ $P\left(1\right)+P\left(-1\right)=2\det A+2\det B\Leftrightarrow\det\left(A+B\right)+\det\left(A-B\right)=2\det A+2\det B.$ $\left\lvert P\left(i\right)\right\rvert^2=P\left(i\right)\cdot P\left(-i\right)=\det\left[\left(A+iB\right)\cdot\left(A-iB\right)\right]=\det\left(A^2+B^2-i\cdot\left(AB-BA\right)\right);$ $\left\lvert P\left(i\right)\right\rvert^2=\det\left[\left(A-iB\right)\cdot\left(A+iB\right)\right]=\det\left(A^2+B^2+i\cdot\left(AB-BA\right)\right);$ $2\cdot\left\lvert P\left(i\right)\right\rvert^2=2\det\left(A^2+B^2\right)+2\det i\left(AB-BA\right)\Rightarrow\left\lvert P\left(i\right)\right\rvert^2=$ $\det\left(A^2+B^2\right)-\det\left(AB-BA\right)\geq 0\Rightarrow\Delta\leq 0.$	**4p**
	$A=\begin{pmatrix}1 & 0\\ 2 & 0\end{pmatrix}$; B= $\begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix}$	**3p**