CONCURSUL INTERJUDEŢEAN DE MATEMATICĂ
" PETRU MOROŞAN -TRIDENT "
Ediţia a X-a , Secţiunea B (M_2),
Brăila, 09 - 11.11. 2012

CLASA a X a

1. a) Demonstraţi identitatea: $\frac{1}{k(k+1)}=\frac{1}{k}-\frac{1}{k+1}, k\in\mathbb{N}^*$

b) Determinaţi produsul rădăcinilor ecuaţiei:

$$\sum_{k=1}^{n}\left(\log_x 2012^{\frac{1}{k}}\right)\cdot\left(\log_x 2012^{\frac{1}{k+1}}\right)+\frac{1}{\left(\log_{2012}x\right)^2}=2-\frac{1}{n+1}, x\in(0,+\infty)-\{1\}.$$

Roxandra Murea

Soluţie:

a) $\frac{1}{k(k+1)}=\frac{1}{k}-\frac{1}{k+1}$

b) Ecuaţia este echivalentă cu:

$$\sum_{k=1}^{n}\frac{1}{k}\cdot\frac{1}{k+1}(\log_x 2012)^2+(\log_x 2012)^2=2-\frac{1}{n+1}\Leftrightarrow(\log_x 2012)^2\left(\sum_{k=1}^{n}\frac{1}{k(k+1)}+1\right)=2-\frac{1}{n+1}$$

Pe de altă parte $\sum_{k=1}^{n}\frac{1}{k(k+1)}=1-\frac{1}{n+1}$ conform punctului a).Ecuaţia devine:

$(\log_x 2012)^2=1\Leftrightarrow x_1=2012, x_2=\frac{1}{2012}$. De unde $P=1$

1.Fie $z_1, z_2 z_3\in\mathbb{C}$ distincte,cu $|z_1|=|z_2|=|z_3|=1, |kz_1-z_2-z_3|=|kz_2-z_1-z_3|=|kz_3-z_1-z_2|$,

$k\in\mathbb{R}/\{-1\}$, atunci $z_1+z_2+z_3=0$.

Gheorghe Alexe, Brăila

***Soluţie*:**

Fie $z=z_1+z_2+z_3\Rightarrow|kz_1-z_2-z_3|=|(k+1)z_1-z|=|(k+1)z_2-z|=|(k+1)z_3-z|$. Ridicăm relaţia la pătrat şiobţinem $|(k+1)z_1-z|^2=|(k+1)z_2-z|^2=|(k+1)z_3-z|^2\Rightarrow\left[(k+1)z_1-z\right]\left[(k+1)\overline{z_1}-\overline{z}\right]=$

$=\left[(k+1)z_2-z\right]\left[(k+1)\overline{z_2}-\overline{z}\right]=\left[(k+1)z_3-z\right]\left[(k+1)\overline{z_3}-\overline{z}\right]\Leftrightarrow\overline{z}=\frac{z}{z_1z_2}$ şi $\overline{z}=\frac{z}{z_2z_3}\Rightarrow\frac{z}{z_1z_2}=\frac{z}{z_2z_3}\Rightarrow$

$\Rightarrow z\cdot z_3=z\cdot z_1\Rightarrow z\cdot\underbrace{(z_3-z_1)}_{\neq 0}\Rightarrow z=0\Rightarrow z_1+z_2+z_3=0.$

3. Fie $a_1, a_2, ...a_n \in (0,1)$ sau $a_1, a_2....a_n \in (1,\infty)$, $n \in \mathbb{N}, n \geq 2$. Demonstraţi inegalităţile:

a) $\log_{a_1} a_2 + 2 \cdot \sqrt{\log_{a_2} a_3} + 3 \cdot \sqrt[3]{\log_{a_3} a_1} \geq 6$

b) $\log_{a_1} a_2 + 2\sqrt{\log_{a_2} a_3} + 3\sqrt[3]{\log_{a_3} a_4} + \cdots + n\sqrt[n]{\log_{a_n} a_1} \geq \frac{n(n+1)}{2}$

Runceanu Emilian

Soluţie:

a) Din inegalitatea mediilor :

$$\frac{\log_{a_1} a_2 + \sqrt{\log_{a_2} a_3} + \sqrt{\log_{a_2} a_3} + \sqrt[3]{\log_{a_3} a_1} + \sqrt[3]{\log_{a_3} a_1} + \sqrt[3]{\log_{a_3} a_1}}{6} \geq \sqrt[6]{\log_{a_1} a_2 \cdot \log_{a_2} a_3 \cdot \log_{a_3} a_1} =$$

$= \sqrt[6]{\log_{a_1} a_2 \cdot \frac{\log_{a_1} a_3}{\log_{a_1} a_2} \cdot \frac{1}{\log_{a_1} a_3}} = 1$. Eliminând numitorul rezultă cerinţa.

b) $$\frac{\log_{a_1} a_2 + \sqrt{\log_{a_2} a_3} + \sqrt{\log_{a_2} a_3} + \sqrt[3]{\log_{a_3} a_4} + \sqrt[3]{\log_{a_3} a_4} + \sqrt[3]{\log_{a_3} a_4} + \cdots + \sqrt[n]{\log_{a_n} a_1} + \cdots + \sqrt[n]{\log a_n a_1}}{1+2+3+\cdots n} \geq$$

$\geq \sqrt[1+2+\cdots n]{\log_{a_1} a_2 \cdot \log_{a_2} a_3 \cdot \log_{a_3} a_4 \cdots \log_{a_n} a_1} = 1.$

Ținând cont că $1+2+3+\cdots+n = \frac{n(n+1)}{2}$ şi eliminând numitorul avem relaţia ce trebuia demonstrată.