Team competition

9th september 2012

6TH MIDDLE EUROPEAN
MATHEMATICAL OLYMPIAD
SOLOTHURN 2012
SWITZERLAND

Language: English

Problem T-1.
Find all triplets (x, y, z) of real numbers such that

$$2x^3 + 1 = 3zx,$$
$$2y^3 + 1 = 3xy,$$
$$2z^3 + 1 = 3yz.$$

Problem T-2.
Let a, b, and c be positive real numbers with $abc = 1$. Prove that

$$\sqrt{9 + 16a^2} + \sqrt{9 + 16b^2} + \sqrt{9 + 16c^2} \geq 3 + 4(a + b + c).$$

Problem T-3.
Let n be a positive integer. Consider words of length n composed of letters from the set $\{M, E, O\}$. Let a be the number of such words containing an even number (possibly 0) of blocks ME and an even number (possibly 0) of blocks MO. Similarly, let b be the number of such words containing an odd number of blocks ME and an odd number of blocks MO. Prove that $a > b$.

Problem T-4.
Let $p > 2$ be a prime number. For any permutation $\pi = (\pi(1), \pi(2), \ldots, \pi(p))$ of the set $S = \{1, 2, \ldots, p\}$, let $f(\pi)$ denote the number of multiples of p among the following p numbers:

$$\pi(1),\ \pi(1) + \pi(2),\ \ldots,\ \pi(1) + \pi(2) + \cdots + \pi(p).$$

Determine the average value of $f(\pi)$ taken over all permutations π of S.

Problem T-5.
Let K be the midpoint of the side AB of a given triangle ABC. Let L and M be points on the sides AC and BC, respectively, such that $\sphericalangle CLK = \sphericalangle KMC$. Prove that the perpendiculars to the sides AB, AC, and BC passing through K, L, and M, respectively, are concurrent.

Problem T-6.
Let $ABCD$ be a convex quadrilateral with no pair of parallel sides, such that $\sphericalangle ABC = \sphericalangle CDA$. Assume that the intersections of the pairs of neighbouring angle bisectors of $ABCD$ form a convex quadrilateral $EFGH$. Let K be the intersection of the diagonals of $EFGH$. Prove that the lines AB and CD intersect on the circumcircle of the triangle BKD.

Problem T-7.
Find all triplets (x, y, z) of positive integers such that

$$x^y + y^x = z^y,$$
$$x^y + 2012 = y^{z+1}.$$

Problem T-8.
For any positive integer n, let $d(n)$ denote the number of positive divisors of n. Do there exist positive integers a and b, such that $d(a) = d(b)$ and $d(a^2) = d(b^2)$, but $d(a^3) \neq d(b^3)$?

Time: 5 hours
Time for questions: 45 min
Each problem is worth 8 points.
The order of the problems does not depend on their difficulty.