Problems and Solutions MEMO 2012

Individual Competition

I1. Find all functions $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that equality

$$f(x + f(y)) = yf(xy + 1)$$

holds for all $x, y \in \mathbb{R}_+$.
($\mathbb{R}_+$ denotes the set of all positive real numbers.)

Solution. Assume that there exists a real number t such that $t > 1$ and $f(t) > 1$. If we put $x = \dfrac{f(t) - 1}{t - 1}$, $y = t$ into the given equality, we get:

$$f(\frac{f(t) - 1}{t - 1} + f(t)) = tf(\frac{f(t) - 1}{t - 1} \cdot t + 1),$$

which can be written as

$$f(\frac{tf(t) - 1}{t - 1}) = tf(\frac{tf(t) - 1}{t - 1}).$$

From the previous equality, we conclude that $t = 1$, which contradicts our assumption that $t > 1$. Therefore, we conclude that such t doesn't exist. In the same way, by putting $x = \dfrac{1 - f(t)}{1 - t}$, $y = t$, we can prove that there doesn't exist a real number $t < 1$ such that $f(t) < 1$.

Let $y > 1$ be an arbitrary real number and let $x = \dfrac{y - 1}{y}$. By plugging them into the given equality, we get:

$$f(\frac{y - 1}{y} + f(y)) = yf(y).$$

If $f(y) > \frac{1}{y}$ then $f(\frac{y-1}{y} + f(y)) > 1$ and thus $\frac{y-1}{y} + f(y) \leqslant 1$. However, the last inequality implies that $f(y) \leqslant \frac{1}{y}$, which is a contradiction. In the same way, if we assume that $f(y) < \frac{1}{y}$, we also get to a contradiction. Therefore, $f(y) = \frac{1}{y}$ for all $y > 1$.

Finally, let $0 < a \leqslant 1$. Let us take an arbitrary y such that $y > \frac{1}{a} \geqslant 1$ and let us denote $x = a - \frac{1}{y}$. By plugging this into the given equality, we get:

$$f(a) = f(x + \frac{1}{y}) = f(x + f(y)) = yf(xy + 1) = y \cdot \frac{1}{xy + 1} = \frac{1}{\frac{xy+1}{y}} = \frac{1}{x + \frac{1}{y}} = \frac{1}{a}.$$

Therefore, the only solution to the given functional equation is $f(x) = \dfrac{1}{x}$.

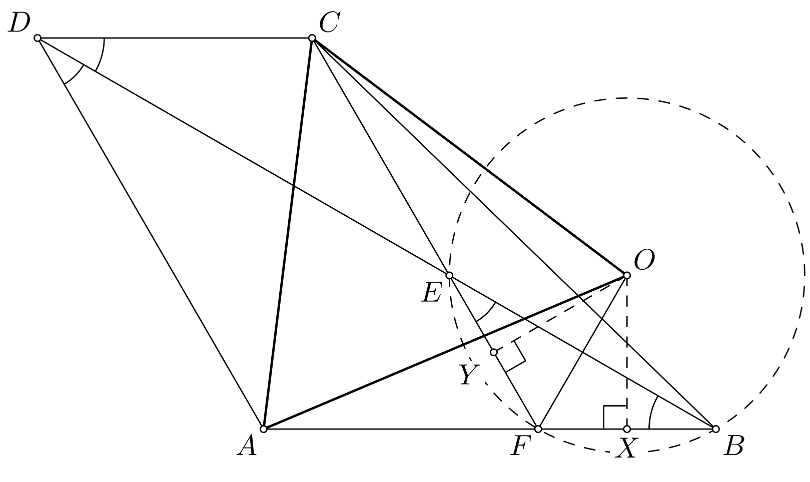
I2. Let N be a positive integer. A set $S \subset \{1, 2, \ldots, N\}$ is called allowed if it does not contain three distinct elements a, b, c such that a divides b and b divides c. Determine the largest possible number of elements in an allowed set S.

Solution. The answer is $\lceil \frac{3}{4}N \rceil$.
On one hand, we can reach this optimum by choosing $S^* = \{\lfloor \frac{1}{4}N \rfloor + 1, \lfloor \frac{1}{4}N \rfloor + 2, \ldots, N\}$. We call a triplet (a, b, c) forbidden if a divides b and b divides c. For any forbidden triplet we would have $4a \leqslant c$ (as if $x|y$ and $x \neq y$ then $2x \leqslant y$), hence there are no forbidden triplets in S^* which means that it is allowed.
On the other hand, there are $\lceil N/2 \rceil$ odd numbers in $\{1, 2, \ldots, N\}$. For such an odd number q, consider the set $H_q = \{q, 2q, 4q, \ldots 2^{i_q} q\}$, where i_q is the largest index i such that $2^i q \leqslant N$. We partitioned $\{1, \ldots, N\}$ to these H_q sets. Any three elements from a single set H_q form a forbidden triplet, hence for any feasible set S we have $|H_q \cap S| \leqslant 2$ (for all q odd numbers). If $q > N/2$, then $|H_q| = 1$. So for all $1 \leqslant q \leqslant N/2$ we can choose maximum 2 elements of H_q, and for all $N/2 < q \leqslant N$ we can choose only 1 element of H_q. In both cases the number is equal to $|H_q \cap S^*|$. In summary we have $|H_q \cap S| \leqslant |H_q \cap S^*|$ for all odd q which implies that $|S| \leqslant |S^*| = \lceil 3N/4 \rceil$.

I3. In a given trapezium $ABCD$ with AB parallel to CD and $AB > CD$, the line BD bisects the angle $\sphericalangle ADC$. The line through C parallel to AD meets the segments BD and AB in E and F, respectively. Let O be the circumcentre of the triangle BEF. Suppose that $\sphericalangle ACO = 60°$. Prove the equality

$$|CF| = |AF| + |FO|.$$

Solution.



Let $|AB| = a$ and $|CD| = b$. Since $AB \parallel CD$ we have

$$\sphericalangle DBA = \sphericalangle BDC = \sphericalangle ADB$$

and thus the triangle ABD is isosceles with $|AB| = |DA| = a$.

Since quadrilateral $AFCD$ is parallelogram, we have

$$|AF| = |CD| = b, \quad |FC| = |DA| = a \quad \text{and} \quad |FB| = a - b.$$

Since $AD \parallel CF$, we have

$$\sphericalangle FEB = \sphericalangle ADB = \sphericalangle DBA$$

and thus the triangle FBE is isosceles with $|FB| = |EF| = a - b$.

Let X and Y be the midpoints of segments $\overline{FB}$ and $\overline{EF}$. Due to symmetry, $|OX| = |OY|$. From

$$|AX| = |AF| + |FX| = b + \frac{a-b}{2} = \frac{a+b}{2}$$

and

$$|CY| = |CF| - |FY| = a - \frac{a-b}{2} = \frac{a+b}{2}$$

we can conclude that right-angled triangles AXO and CYO are congruent and $|AO| = |CO|$. Since $\sphericalangle ACO = 60^\circ$, the triangle AOC is equilateral and $\sphericalangle OAC = 60^\circ$.

Denoting $\sphericalangle XAO = \sphericalangle YCO = \varphi$ we get $\sphericalangle FAC = \sphericalangle XAO + \sphericalangle OAC = \varphi + 60^\circ$ and $\sphericalangle ACF = \sphericalangle ACO - \sphericalangle YCO = 60^\circ - \varphi$, so $\sphericalangle CFA = 60^\circ$, $\sphericalangle BFE = 120^\circ$ and $\sphericalangle XFO = 60^\circ$.

Since the angles of the triangle FXO are 30°, 60° and 90°, from $|FO| = 2|FX| = a - b$, we finally get $|AF| + |FO| = b + (a - b) = a = |CF|$.

I4. The sequence $\{a_n\}_{n\geqslant 0}$ is defined by $a_0 = 2$, $a_1 = 4$ and

$$a_{n+1} = \frac{a_n a_{n-1}}{2} + a_n + a_{n-1} \text{ for all } n > 0.$$

Determine all prime numbers p for which there exist $m > 0$ such that $p | a_m - 1$.

Solution. If a_{n-1} and a_n are even the number $a_n a_{n-1}/2$ is even and therefore a_{n+1}, too. Hence a_n is even for all $n > 0$ and $p = 2$ is not a solution. Since $3 | a_1 - 1$, $p = 3$ is a solution. Assume from now on $p \geqslant 5$. The equation is equivalent to

$$a_{n+1} + 2 = \frac{(a_n + 2)(a_{n-1} + 2)}{2}.$$

Let $b_n = (a_n + 2)/2$, it follows that $b_{n+1} = b_n b_{n-1}$ with $b_1 = 2$ and $b_2 = 3$. Note that $p \nmid b_n$ for all $n \in \mathbb{N}$. Hence the sequence modulo p can be constructed backwards and extended for negative numbers n by $b_n \equiv \frac{b_{n+2}}{b_{n+1}}$ modulo p. Note that two consecutive elements b_k, b_{k+1} determine the entire sequence. As there are p^2 ordered pairs of residues modulo p it follows by the pigeon hole principle that there exists $0 \leqslant k < l \leqslant p^2$ such that $b_k \equiv b_l$ and $b_{k+1} \equiv b_{l+1}$. The sequence modulo p is therefore periodic with length $l - k$ and we have $b_{l-k-1} \equiv b_{-1} \equiv \frac{p+3}{2}$. It follows that $a_{l-k-1} \equiv 1$ modulo p. Hence all prime numbers greater than 2 are solutions.

Team Competition

T1. Find all triplets (x, y, z) of real numbers such that

$$\begin{aligned} 2x^3+1 &= 3zx, \\ 2y^3+1 &= 3xy, \\ 2z^3+1 &= 3yz. \end{aligned}$$

Solution. Note that no variable can be zero. Assume that one variable is positive. WLOG $x > 0$. It follows that $z > 0$ and $y > 0$. Summing up all three equations and using AM-GM yields

$$2(x^3+y^3+z^3)+3 = 3(xy+yz+zx) \leqslant 3(\frac{x^3+y^3+1}{3}+\frac{y^3+z^3+1}{3}+\frac{z^3+x^3+1}{3}) = 2(x^3+y^3+z^3)+3.$$

Therefore we must have equality and it follows $x = y = z = 1$ which is a solution.
Assume from now on that all variables are negative. Put $u = -x, v = -y$ and $w = -z$. Then u, v, w must be positive and the equations can then be written as

$$\begin{aligned} 1 &= 2u^3+3uw, \\ 1 &= 2v^3+3vu, \\ 1 &= 2w^3+3wv. \end{aligned}$$

WLOG let $u \geqslant v, w$. We have

$$1 = 2u^3+3uw \geqslant 2w^3+3wv = 1.$$

Hence we must have equality and therefore $u = v = w$. It follows that

$$0 = 2u^3+3u^2-1 = (u+1)^2(2u-1).$$

Hence $v = w = u = 1/2$ which is also a solution. The system has therefore the solutions $(x, y, z) = (1, 1, 1)$ and $(x, y, z) = (-1/2, -1/2, -1/2)$.

T2. Let a, b and c be positive real numbers with $abc = 1$. Prove that

$$\sqrt{9+16a^2}+\sqrt{9+16b^2}+\sqrt{9+16c^2} \geqslant 3+4(a+b+c).$$

First Solution. Let us start by showing

(⊠) *If three positive reals x, y, and z sum up to less than 3, then*

$$\frac{9-x^2}{x}\cdot\frac{9-y^2}{y}\cdot\frac{9-z^2}{z}>512.$$

Proof of (⊠). Using the inequality between the arithmetic mean and the geometric mean of four positive reals we obtain

$$3+x=1+1+1+x\geqslant 4\sqrt[4]{x}.$$

For similar reasons, one has $3+y\geqslant 4\sqrt[4]{y}$ and $3+z\geqslant 4\sqrt[4]{z}$. Multiplying the three foregoing estimates we get

$$(3+x)(3+y)(3+z)\geqslant 64\sqrt[4]{xyz}. \qquad (1)$$

Using the A.M.–G.M. inequality with just three variables, we get

$$3>x+y+z\geqslant 3\sqrt[3]{xyz},$$

which tells us

$$1>xyz,$$

as well as

$$xy+yz+zx\geqslant 3(xyz)^{2/3}.$$

Combining these estimates we infer

$$\begin{aligned}(3-x)(3-y)(3-z)&=9(3-x-y-z)+3(xy+yz+zx)-xyz\\&>9(xyz)^{2/3}-xyz>8(xyz)^{2/3}.\end{aligned}$$

If we finally multiply this by (1), we arrive at

$$(9-x^2)(9-y^2)(9-z^2)>512xyz$$

and (⊠) follows.

Now to attack the problem itself, we define three positive reals x, y, and z by

$$\begin{aligned}x&=\sqrt{9+16a^2}-4a,\\y&=\sqrt{9+16b^2}-4b,\\\text{and}\quad z&=\sqrt{9+16c^2}-4c.\end{aligned}$$

Then

$$\frac{9-x^2}{x}\cdot\frac{9-y^2}{y}\cdot\frac{9-z^2}{z}=512abc=512,$$

which means that the conclusion of (⊠) is violated. This is only possible if its assumption does not hold, i.e. if $x+y+z\geqslant 3$. Hence

$$\sqrt{9+16a^2}+\sqrt{9+16b^2}+\sqrt{9+16c^2}\geqslant 3+4(a+b+c)$$

and the problem is solved.

Second Solution. The function $x \mapsto e^x$ where e is the Euler Constant is a bijection between $\mathbb{R}$ and $\mathbb{R}_{>0}$, hence we can find $u, v, w \in \mathbb{R}$ such that $a = e^u, b = e^v$ and $c = e^w$ and the constraint $abc = 1$ becomes

$$e^u e^v e^w = 1 \Leftrightarrow e^{u+v+w} = 1 \Leftrightarrow u + v + w = 0.$$

Let $f(x) = \sqrt{e^{2x} + \frac{9}{16}} - e^x$ the inequality can then be written as

$$f(u) + f(v) + f(w) \geqslant \frac{3}{4} \qquad \forall u, v, w \text{ with } u + v + w = 0$$

With some calculus we compute the first two derivatives of f

$$f'(x) = \frac{e^{2x}}{\sqrt{e^{2x} + \frac{9}{16}}} - e^x \tag{2}$$

$$f''(x) = \frac{2e^{2x}(e^{2x} + \frac{9}{16}) - e^{4x}}{(e^{2x} + \frac{9}{16})^{\frac{3}{2}}} - e^x \tag{3}$$

$$= \frac{e^{2x}(e^{2x} + \frac{9}{8})}{(e^{2x} + \frac{9}{16})^{\frac{3}{2}}} - e^x. \tag{4}$$

We determine the convex intervals of f:

$$f''(x) > 0 \tag{5}$$

$$\Leftrightarrow e^{2x}(e^{2x} + \frac{9}{8}) > e^x(e^{2x} + \frac{9}{16})^{\frac{3}{2}} \tag{6}$$

$$\Leftrightarrow e^{4x}(e^{2x} + \frac{9}{8})^2 > e^{2x}(e^{2x} + \frac{9}{16})^3 \tag{7}$$

$$\Leftrightarrow \frac{9}{16}e^{4x} + \frac{81}{256}e^{2x} - \left(\frac{9}{16}\right)^3 > 0 \tag{8}$$

As the function $g(x) = \frac{9}{16}e^{4x} + \frac{81}{256}e^{2x} - \left(\frac{9}{16}\right)^3$ is monotone increasing, $\lim_{x\to-\infty} g(x) = -\left(\frac{9}{16}\right)^3$ and $\lim_{x\to\infty} g(x) = \infty$ there is exactly one value $m \in \mathbb{R}$ with $g(m) = 0$. Furthermore it follows that f is convex on (m, ∞) and concave on $(-\infty, m)$.

Lemma 1. *For any three real numbers x_1, x_2, x_3 with $x_1 + x_2 + x_3 = 0$ there exists real numbers u, v with $u + 2v = 0$ such that*

$$f(x_1) + f(x_2) + f(x_3) \geqslant f(u) + 2f(v).$$

Proof. WLOG we can assume $x_1 \leqslant x_2 \leqslant x_3$. If $x_2 < m$ we have $f(x_1) + f(x_2) \geqslant f(x_1 + x_2 - m) + f(m)$ as f is concave on $(-\infty, m)$. Hence we can assume $x_2 \geqslant m$. Then we have by Jensen's inequality $f(x_2) + f(x_3) \geqslant 2f(\frac{x_2+x_3}{2})$. Hence we can set $u = x_1$ and $v = \frac{x_2+x_3}{2}$. □

It remains to prove that

$$2\sqrt{a^2 + \frac{9}{16}} + \sqrt{b^2 + \frac{9}{16}} \geqslant 2a + b + \frac{3}{4} \qquad \text{for all } a, b \text{ with } a^2 b = 1. \tag{9}$$

By squaring and rearranging the inequality becomes

$$4\sqrt{(a^2 + \frac{9}{16})(b^2 + \frac{9}{16})} \geqslant 4ab + 3a + \frac{3}{2}b - \frac{9}{4}.$$

Dividing by 4, squaring and rearranging yields

$$\frac{27}{64}b^2 + \frac{9}{16}ab + \frac{27}{32}a + \frac{27}{64}b \geqslant \frac{3}{2}a^2b + \frac{3}{4}ab^2.$$

After multiplying by a^4 and using $a^2b = 1$ we arrive at

$$\frac{27}{32}a^5 - \frac{3}{2}a^4 + \frac{9}{16}a^3 + \frac{27}{64}a^2 - \frac{3}{4}a + \frac{27}{64} \geqslant 0 \quad (10)$$

$$\Leftrightarrow (a-1)^2(\frac{27}{32}a^3 + \frac{3}{16}a^2 + \frac{3}{32}a + \frac{27}{64}) \geqslant 0 \quad (11)$$

which is obviously true.

T3. Let n be a positive integer. Consider words of length n composed of letters from the set $\{M, E, O\}$. Let a be the number of such words containing an even number (possibly 0) of blocks ME and an even number (possibly 0) of blocks MO. Similarly, let b be the number of such words containing an odd number of blocks ME and an odd number of blocks MO. Prove that $a > b$.

First Solution. Let A be the set of words of length n with even number of ME and even number of MO and B be the set of words of length n with odd number of ME and odd number of MO. We construct an injective map f from B to A. Choose the first place in the tuple, where either is the block 01 or the block 02 (this place must exist since the number of 01-blocks is odd and therefore >0) and interchange the blocks. One of the number of blocks 01 and the number of blocks 02 will increase by 1 and one will decrease by 1. Hence both numbers change its parity from odd to even and the image of the map is therefore in A. Furthermore when the operation above is applied two times on a sequence the we get the original sequence. Hence we have $f(f(b)) = b$ for all $b \in B$ which implies that the operation is injective and therefore $|B| \leqslant |A|$. To see that $|B| < |A|$ note that all sequences which do not contain an 0 are in A but can not be the image of an element of B under f. As $n > 0$ there is at least one such sequence.

Second Solution. We call the blocks ME and the blocks MO important and we say that two words are similar if the positions of the important blocks are the same. This similarity divides the set of all sequences into equivalence classes. Let A and B be defined as in the first solution. Note that the elements of A and the elements of B contain an even number of important blocks. Therefore an equivalence class with an odd number of important blocks does not contain any elements of A or B. Consider an equivalence class with $k \geqslant 2$ even number of important blocks. There are $n_{\text{even}} = \sum_{m \text{ even}} \binom{k}{m}$ elements of the equivalence class in A and $n_{\text{odd}} = \sum_{m \text{ odd}} \binom{k}{m}$ elements of the equivalence class in B. We have

$$n_{\text{even}} - n_{\text{odd}} = (1-1)^k = 0.$$

which proves that the equivalence class contains the same number of elements from A and B. For $k = 0$ all elements belong to A and this set is not empty which concludes our proof.

Third Solution. Let a_n be the number of words of length n with even number of ME and even number of MO, b_n be the number of words of length n with odd number of ME and odd number of MO and x_n be the common number of words such that

1) start with E, odd number of ME blocks, even number of MO blocks
2) start with O, odd number of ME blocks, even number of MO blocks
3) start with E, even number of ME blocks, odd number of MO blocks
4) start with O, even number of ME blocks, odd number of MO blocks.

Bijections 1) $\leftrightarrow$ 4) and 2) $\leftrightarrow$ 3) : replace all E by O and all O by E
Bijections 1) $\leftrightarrow$ 2) and 3) $\leftrightarrow$ 4) : switch first number E $\leftrightarrow$ O

To compute a_{n+1} we consider the three possible start letters of the word. If it starts with E or O the number of words is a_n. If it starts with M we distinguish again the three cases for the second letter. If the word starts with ME or MO we get x_n words. When the sequence starts with MM we distinguish again three cases.... If we continue this argument we arrive at

$$a_{n+1} = 2a_n + 2\sum_{k=1}^{n} x_k + 1 \tag{12}$$

(the plus 1 comes from the sequence $M \dots M$).
Similarly

$$b_{n+1} = 2b_n + 2\sum_{k=1}^{n} y_k \tag{13}$$

Taking differences:

$$a_{n+1} - b_{n+1} = 2(a_n - b_n).$$

Since $a_1 = 3$ and $b_1 = 0$ an induction gives the claim

$$a_n > b_n.$$

(Actually $a_n = b_n + 2^{n+1} - 1$)

T4. Let $p > 2$ be a prime number. For any permutation $\pi = (\pi(1), \pi(2), \dots, \pi(p))$ of the set $S = \{1, 2, \dots, p\}$, let $f(\pi)$ denote the number of multiples of p among the following p numbers:

$$\pi(1),\ \pi(1) + \pi(2), \dots, \pi(1) + \pi(2) + \dots + \pi(p)$$

What is the average value of $f(\pi)$ taken over all permutations of S?

Solution. We call two permutations π' and π'' equivalent if there is an integer d such that:

$$\pi'(i) + d \equiv \pi''(i) \pmod p$$

for all $i = 1, 2, \ldots, p$. (It's trivial to check that it is an equivalence relation). This equivalence partitions the set $\mathcal{P}$ of all permutations into equivalence classes. Each class consists of p permutations, that can be labeled $\pi_0, \ldots, \pi_d, \ldots, \pi_{p-1}$ such that $\pi_0(i) + d$ and $\pi_d(i)$ are congruent for all i.

Let s_k denote the sum $\pi_0(1) + \pi_0(2) + \cdots + \pi_0(k)$ mod p. Then $\pi_d(1) + \pi_d(2) + \cdots + \pi_d(k) \equiv s_k + kd \pmod p$. So for a fixed k if we consider the sums $\pi_d(1) + \pi_d(2) + \cdots + \pi_d(k)$ for all d-s, we get these:

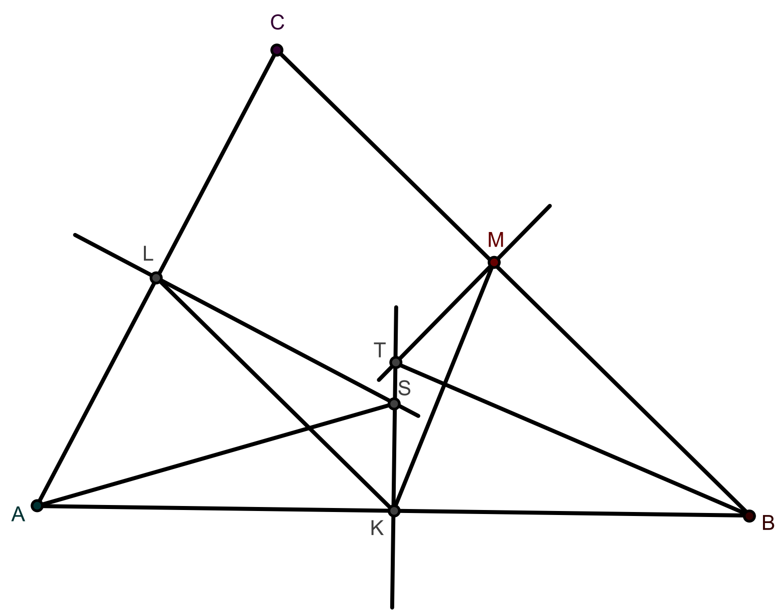
$$s_k, s_k + k, s_k + 2k, \ldots, s_k + (p-1)k$$

If $k < p$ then among these there is exactly one 0. If $k = p$ then all are 0s (s_p is the sum of all remainder classes mod p, it is a well-known fact that it's zero if $p > 2$).

So for any equivalence class, consisting of p pieces of permutations, there $2p-1$ zeros among the sums assigned to them. That means an average of $\frac{2p-1}{p}$.

T5. Let K be the midpoint of the side AB of a given triangle ABC. Let L and M be points on the sides AC and BC, respectively, such that $\sphericalangle CLK = \sphericalangle KMC$. Prove that perpendiculars to the sides AB, AC and BC passing through K, L and M, respectively, are concurrent.

Solution.



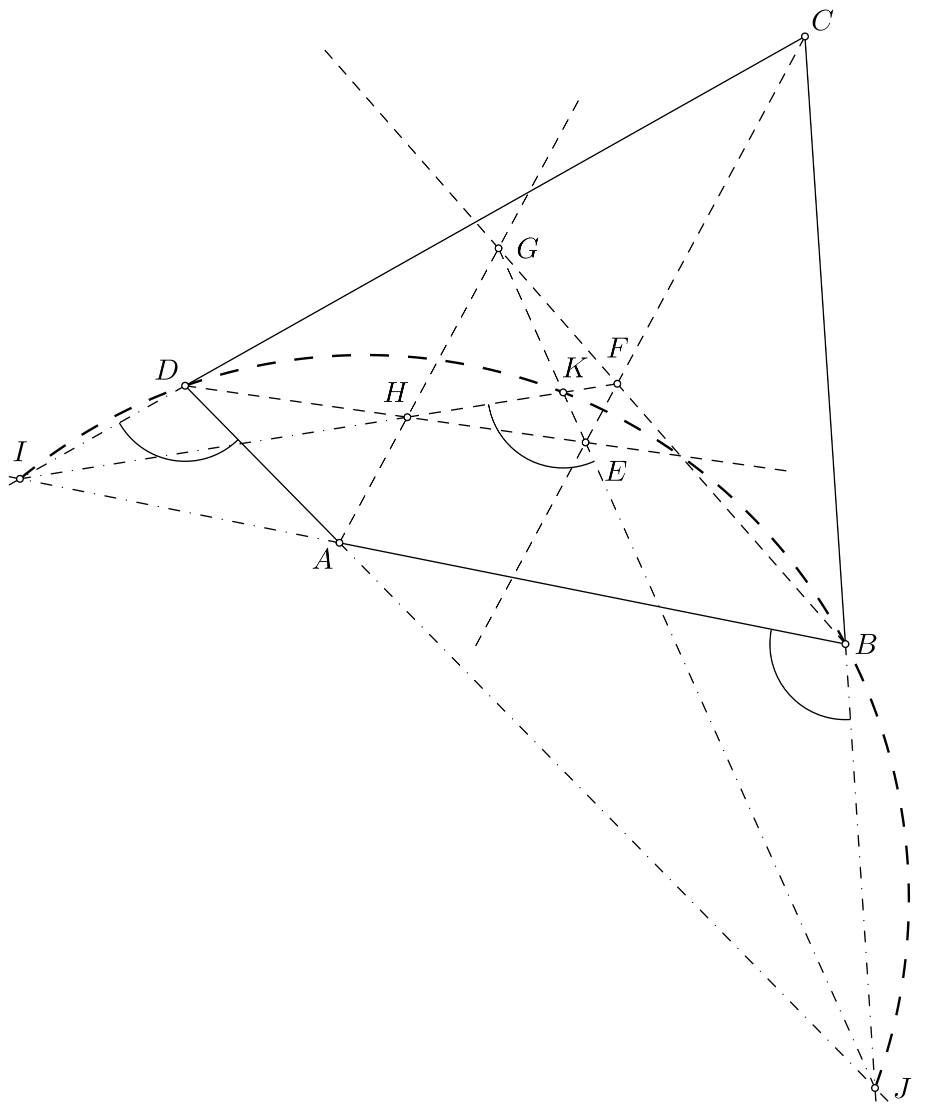
Let S be the intersection of perpendiculars through K and L, and T the intersection of perpendiculars through K and M. Observe that $AKSL$ and $BMTM$ are cyclic (they have two right angles). Therefore $\sphericalangle SAK = \sphericalangle SLK = \sphericalangle CLK - \frac{\pi}{2}$ (or $\frac{\pi}{2} - \sphericalangle CKL$ if $\sphericalangle CKL$ is acute). Analogously, $\sphericalangle TBK = \sphericalangle TMK = \sphericalangle CMK - \frac{\pi}{2}$ (or $\frac{\pi}{2} - \sphericalangle CMK$ if $\sphericalangle CMK$ is acute). Therefore

$$\sphericalangle SAK = \sphericalangle TBK.$$

But K is the midpoint of AB, hence S and T have to coincide. Thus the perpendiculars are concurrent.

T6. Let $ABCD$ be a convex quadrilateral with no pair of parallel sides, such that $\sphericalangle ABC = \sphericalangle CDA$. Assume that the intersections of the pairs of neighbouring angle bisectors of $ABCD$ form a convex quadrilateral $EFGH$. Let K be the intersection of the diagonals of $EFGH$. Prove that the lines AB and CD intersect on the circumcircle of the triangle BKD.

Solution.



Let's denote $I \equiv AB \cap CD$ and $J \equiv BC \cap AD$. Now F is the incenter of triangle IBC thus IF is the angle bisector of $\sphericalangle BIC$. We can also note that H is the excenter of triangle ADI su IH is the same angle bisector. Thus I, F, H are collinear. similarly we can deduce that J, E, G are collinear as well. Now if we denote $\alpha = \sphericalangle BAC$ and $\beta = \sphericalangle ABC = \sphericalangle CDA$ we can see that $\sphericalangle JDI = 180° - \beta = \sphericalangle JBI$. Since IK and JK are the angle bisectors of $\sphericalangle DIA$ and $\sphericalangle AJB$ we get:

$$\begin{aligned}
\sphericalangle IKJ &= 360° - \sphericalangle KIA - \sphericalangle KJA - \sphericalangle JAI \\
&= 360° - \frac{\sphericalangle AID}{2} - \frac{\sphericalangle AJB}{2} - (360° - \alpha) \\
&= \alpha - \frac{\alpha + \beta - 180°}{2} - \frac{\alpha + \beta - 180°}{2} \\
&= 180° - \beta = \sphericalangle JDI = \sphericalangle JBI.
\end{aligned}$$

Thus the points I, J, B, K, D are all concyclic which implies the desired result.

T7. Find all triplets (x, y, z) of positive integers such that

$$\begin{cases} x^y + y^x = z^y, \\ x^y + 2012 = y^{z+1}. \end{cases}$$

Solution. We consider two cases.

(i) Let x be an odd number. Then, clearly, y is odd (the second equation) and z is even (the first equation). If a and b are odd positive integers then $a^b \equiv (\pm 1)^b \equiv \pm 1 \equiv a \pmod 4$. Let us apply this fact to both equations. The second equation implies that $y > 1$, thus, 4 divides z^y and $x + y \equiv x^y + y^x \equiv z^y \equiv 0 \pmod 4 \implies x \equiv -y \pmod 4$. On the other hand, $x \equiv x + 0 \equiv x^y + 2012 \equiv y^{z+1} \equiv y \pmod 4$. This means that $y \equiv -y \pmod 4$ which is impossible for an odd y.

(ii) Let $x = 2x_1$ be even. Then y and z are even too. If $y > 2$ then 8 divides x^y, but only 4 divides 2012, thus, 8 does not divide y^{z+1}. This implies that $z + 1 \leqslant 2 \implies z = 1$ which is impossible in the light of the first equation. Hence, $y \leqslant 2$. We already noted that $y > 1$, thus, $y = 2$. We rewrite the first equation:

$$x^2 + 2^x = z^2 \implies 2^x = (z - x)(z + x).$$

Hence, as x and z are both even, there exist such positive integers $u < v$ that $z - x = 2^u, z + x = 2^v$ and $u + v = x$. Since $v > u$ we have $v \geqslant \frac{x}{2}$ and $u \leqslant v - 1$. This implies that $x = 2^{v-1} - 2^{u-1} \geqslant 2^{v-1} - 2^{v-2} = 2^{v-2} \geqslant 2^{\frac{x}{2}-2} \implies x_1 \geqslant 2^{x_1-3}$. The inequality does not hold for $x_1 = 6$ and the same can be easily proved by induction for the larger values of x_1 (as the left-hand side increases by 1, the right-hand side increases by $2^{x_1-3} > 2^2 > 1$). Hence, we only need to check the first five values $1, 2, 3, 4, 5$ (i.e. $x = 2, 4, 6, 8, 10$). Only for $x = 6$ the number $z = \sqrt{x^2 + 2^x} = 10$ is an integer. The obtained triplet $(6, 2, 10)$ satisfies both equations.

The answer. $(6, 2, 10)$.

T8. For any positive integer n, let $d(n)$ denote the number of positive divisors of n. Do there exist positive integers a and b, such that $d(a) = d(b)$ and $d(a^2) = d(b^2)$, but $d(a^3) \neq d(b^3)$?

First Solution. The answer is negative.
Denote by $\tau(n)$ the number of divisors of $n \in \mathbb{N}$. First note that if the prime factorization of n

is $n = p_1^{a_1} \cdot p_2^{a_2} \cdots p_k^{a_k}$, then $\tau(n) = (a_1+1)\cdots(a_k+1)$ and similarly we find that $\tau(n^2) = (2a_1+1)\cdots(2a_k+1)$ and $\tau(n^3) = (3a_1+1)\cdots(3a_k+1)$. For convenience denote $\beta_i = \alpha_i + 1$ for $i \in \{1, 2, \cdots, k\}$. Then

$$\tau(n) = \prod_{i=1}^{k} \beta_i, \tau(n^2) = \prod_{i=1}^{k} (2\beta_i - 1), \tau(n^3) = \prod_{i=1}^{k} (3\beta_i - 2)$$

Now we offer two approaches to find a counterexample in terms of the numbers β_i for $i \in \{1, 2, \cdots, k\}$.

We look at pairs of positive integers of the form $(n, 2n-1)$ namely at $(2,3), (3,5), (4,7), (8,15), (18,35), (32,63)$ and we aim to find numbers $a_1, a_2, \cdots a_6 \in \mathbb{Z}$ (possibly negative) such that

$$2^{a_1} \cdot 3^{a_2} \cdot 4^{a_3} \cdot 8^{a_4} \cdot 18^{a_5} \cdot 32^{a_6} = 1 \text{ and } 3^{a_1} \cdot 5^{a_2} \cdot 7^{a_3} \cdot 15^{a_4} \cdot 35^{a_5} \cdot 63^{a_6} = 1.$$

Comparing the primes powers of 2, 3, 5 and 7 in the latter relations produces a system of equations

$$a1 + 2a3 + 3a4 + a5 + 5a6 = 0 \quad (14)$$
$$a2 + 2a5 = 0 \quad (15)$$
$$a1 + a4 + 2a6 = 0 \quad (16)$$
$$a2 + a4 + a5 = 0 \quad (17)$$
$$a3 + a5 + a6 = 0 \quad (18)$$
$$(19)$$

with solution $(a_1, a_2, a_3, a_4, a_5, a_6) = (1, -2, 0, 1, 1, -1)$. Then we choose $\beta_1 = 2, \beta_2 = 8, \beta_3 = 18, \beta_1' = \beta_2' = 3, \beta_3' = 32$. We have ensured $\prod \beta_i = \prod \beta_i'$ and $\prod(2\beta_i - 1) = \prod(2\beta_i' - 1)$. It remains to see that $11 | \prod(3\beta_i - 2)$ and $11 \nmid \prod(3\beta_i - 2)$, thus the numbers are distinct and we can find the counterexample to the original problem for example as

$$a = 2 \cdot 3^7 \cdot 5^{17}, \; b = 2^2 \cdot 3^2 \cdot 5^{31}.$$

We look for counterexample in the form $\beta_1 = pq, \beta_2 = r, \beta_3 = s$ and $\beta_1' = p, \beta_2' = q, \beta_3' = rs$ for some $p, q, r, s \in \mathbb{N}$. In order to fulfill the second condition we need

$$\frac{(2p-1)(2q-1)}{2pq-1} = \frac{2r-1)(2s-1)}{2rs-1}$$

Thus, looking at the expression $\frac{(2n-1)(2m-1)}{2mn-1}$ for positive integers m and n, we need it to attain the same value twice for two distinct choices of n and m. After subtracting 2 and switching the sign, we obtain

$$\frac{2m+2n-3}{2mn-1}$$

.

We attempt the common value to be of the form $1/k$ (the expression seems asymptotically small) for some $k \in \mathbb{N}$. After some manipulation, the condition turns into

$$(m-k)(n-k) = \frac{1}{2}(2k-1)(k-1)$$

In order to ensure the right-hand side is not a prime, we choose $k = 5$. Then we need to solve

$$(m-5)(n-5) = 18$$

We find two solutions (6, 23), (7, 14) and so we choose $p = 6, q = 23, r = 7, s = 14$. We continue similarly as in first approach and find counterexample

$$a = 2^{137} \cdot 3^{6} \cdot 5^{13},\ b = 2^{5} \cdot 3^{22} \cdot 5^{97}.$$

It can be easily seen that the counterexample needs to have at least three prime factors. Also, the set of pairs (2, 3), (3, 5), (4, 7), (8, 15), (18, 35), (32, 63) from the first approach is the minimal (with smallest maximal element) set for which the system of equations has non-trivial solutions. This leads to the claim that the counterexample from the first approach is also minimal in this sense.

Second Solution. The problem is immediately reduced to finding finite positive integer sequences (x_i) and (y_j) satisfying

$$\prod_i x_i = \prod_j y_j \tag{20}$$

$$\prod_i (2x_i - 1) = \prod_j (2y_j - 1) \tag{21}$$

$$\prod_i (3x_i - 2) \neq \prod_j (3y_j - 2) \tag{22}$$

and such that (x_i) is not a permutation of (y_j).

We will look for a parametric family of solutions. We base our construction on choosing the y's such that the generic expression $2y-1$ is decomposable and the decomposition factors can be recombined to generate the x's in (21). The simplest identity of this type seems to be obtained for $y = 2z^2$,

$$2y - 1 = 4z^2 - 1 = (2z-1)(2z+1) = (2z-1)(2(z+1)-1).$$

We therefore pick now $y_0 = 2z^2$, $y_1 = 2(z+1)^2, \ldots, y_n = 2(z+n)^2$ for some $n \in \mathbb{N}$. With this choice it follows that

$$\begin{aligned}
\prod_{j=0}^{n} (2y_j - 1) &= \prod_{j=0}^{n} (4(z+j)^2 - 1) \\
&= \prod_{j=0}^{n} (2(z+j)-1)(2(z+j+1)-1) \\
&= (2z-1)(2(z+n+1)-1) \prod_{j=1}^{n} (2(z+j)-1)^2.
\end{aligned}$$

The obvious choice now for the sequence (x_i) consists in the set of values z, $z+n+1$, plus two copies of each number in the sequence $(z+j)$ for $j = 1 : n$. Note that in this way there are twice as many x's as y's: $2n+2$ vs. $n+1$. Explicitly, $x_0 = z, x_1 = x_2 = z+1, x_3 = x_4 = z+2, \ldots, x_{2n-1} = x_{2n} = z+n, x_{2n+1} = z+n+1$. With this choice for the sequence (x_i) condition (21) is fulfilled by construction. But what about (20) or (22)?

Let's start with (20) and calculate

$$\prod_i x_i = z(z+n+1)(z+1)^2(z+2)^2 \cdots (z+n)^2,$$

$$\prod_j y_j = 2^{n+1} z^2 (z+1)^2 \cdots (z+n)^2$$

to obtain

$$\prod_j y_j / \prod_i x_i = 2^{n+1} z/(z+n+1).$$

Although the r.h.s. above can not be made equal to 1 as desired, it can be further simplified by choosing $z = n+1$. With this choice we arrive at

$$\prod_j y_j / \prod_i x_i = 2^n. \tag{23}$$

Recall that this holds for our current choice $y_0 = 2n^2, y_1 = 2(n+1)^2, \ldots, y_n = 2(2n)^2$ and $x_0 = n+1, x_1 = x_2 = n+2, x_3 = x_4 = n+3, \ldots, x_{2n-1} = x_{2n} = 2n+1, x_{2n+1} = 2n+2$.

Call now the entire sequence construction above $\mathcal{C}_n$, and consider for arbitrary positive integers n, m the constructions $\mathcal{C}_n$, $\mathcal{C}_m$ as well as $\mathcal{C}_{n+m}$, together with the corresponding sequences $(x_i, y_j), (x'_i, y'_j)$ and (x''_i, y''_j) respectively.

Consider now the sequences (X_i) and (Y_j) obtained by collecting (concatenating) the sequences $(x_i), (x'_i), (y''_j)$ and $(y_j), (y'_j), (x''_i)$ respectively. We claim that (X_i) and (Y_j) are in general sequences that satisfy all three conditions (20), (21), (22). Due to the obvious identity $2^m 2^n = 2^{m+n}$ we deduce from 23 that the sequences (X_i) and (Y_j) satisfy, beside (21), also condition (20).

It remains to investigate condition (22). There are several ways to do this, for example by analyzing prime decompositions of the l.h.s. and r.h.s. respectively (and formulating a condition on m, n such that the largest prime divisor of the l.h.s. does not divide the r.h.s. anymore). Here however we take a different route and estimate the asymptotic behavior of the two sides of equation (22). To this end we use (20) and write

$$\prod_j (3Y_j - 2) / \prod_i (3X_i - 2) = \left(\prod_j (3Y_j - 2) / \prod_j Y_j \right) \times \left(\prod_i X_i / \prod_i (3X_i - 2) \right). \tag{24}$$

To estimate the two factors on the r.h.s. of (24) we first observe that in the generic construction $\mathcal{C}_n$,

$$\prod_j (3y_j - 2)/y_j = 3^{n+1} \prod_{j=0}^{n} (1 - 2/3y_j) = 3^{n+1} A_n, \tag{25}$$

with $A_n \to 1$ as $n \to \infty$, since $\sum_{j=0}^{n} 1/6(n+j)^2 \to 0$ as $n \to \infty$.

We note now that a similar result holds for the sequence (x_i) in construction $\mathcal{C}_n$,

$$\prod_i (3x_i - 2)/x_i = 3^{2(n+1)} \prod_{i=0}^{2n+1} (1 - 2/3x_i) = 3^{2(n+1)} B_n \tag{26}$$

with $B_n \to 2^{-4/3}$ as $n \to \infty$, since $\sum_{i=0}^{2n+1} 2/3x_i \to 4\ln(2)/3$ as $n \to \infty$ due to $\sum_{j=1}^{n} 1/(n+j+a)$ converging to $\ln(2)$ for any real a.

Using the two facts (25), (26) in (24) and in the context of the construction of (X_i, Y_j) via $\mathcal{C}_n, \mathcal{C}_m, \mathcal{C}_{n+m}$ we obtain

$$\begin{aligned}\prod_j (3Y_j - 2)/\prod_i (3X_i - 2) &= 3^{n+1} A_n \cdot 3^{m+1} A_m \cdot 3^{2(n+m+1)} B_{n+m} \cdot \\ &\quad \cdot 3^{-2(n+1)} B_n^{-1} \cdot 3^{-2(m+1)} B_m^{-1} \cdot 3^{-(n+m+1)} A_{n+m}^{-1} \\ &= (1/3) A_n A_m A_{n+m}^{-1} B_n^{-1} B_m^{-1} B_{n+m}. \end{aligned} \tag{27}$$

Since the limit as $n, m \to \infty$ equals $2^{4/3}/3 \neq 1$ we obtain that the desired conclusion (22) holds for m, n large enough and the proof is complete.