Călăraşi 2012 — Danube Cup

Problem 1. Given a positive integer n, determine the maximum number of lattice points in the plane a square of side length $n + \frac{1}{2n+1}$ may cover.

Problem 2. Let ABC be an acute triangle and let A_1, B_1, C_1 be points on the sides BC, CA and AB, respectively. Show that the triangles ABC and $A_1B_1C_1$ are similar ($\angle A = \angle A_1$, $\angle B = \angle B_1$, $\angle C = \angle C_1$) if and only if the orthocentre of the triangle $A_1B_1C_1$ and the circumcentre of the triangle ABC coincide.

Problem 3. Let p and q, $p < q$, be two primes such that $1 + p + p^2 + \cdots + p^m$ is a power of q for some positive integer m, and $1 + q + q^2 + \cdots + q^n$ is a power of p for some positive integer n. Show that $p = 2$ and $q = 2^t - 1$, where t is prime.

Problem 4. Given a positive integer n, show that the set $\{1, 2, \cdots, n\}$ can be partitioned into m sets, each with the same sum, if and only if m is a divisor of $n(n + 1)/2$ which does not exceed $(n + 1)/2$.

Time allowed: $3\frac{1}{2}$ hours.

Each problem is worth 7 points.