Solutions

Problem 1. The positive integer N is said *amiable* if the set $\{1,2,\dots,N\}$ can be partitioned into pairs of elements, each pair having the sum of its elements a perfect square. Prove there exist infinitely many amiable numbers which are themselves perfect squares.

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Solution. Obviously an amiable number N must be even (for the partition into pairs to be possible). Since there will be $N/2$ pairs, but only less than $\sqrt{2N}$ possible squares as sums of elements in pairs, many of the sums will have to be equal. This suggests trying to manufacture some odd a with $1 < a^2 < N$, for which our set

$$\{1,2,\dots,a^2-2,a^2-1,a^2,a^2+1,\dots,N-1,N\}$$

can be partitioned as

$$\{1,a^2-1\}\cup\{2,a^2-2\}\cup\cdots\cup\{a^2,N\}\cup\{a^2+1,N-1\}\cup\cdots$$

with the sum of elements in the first family of pairs being

$$1+(a^2-1)=2+(a^2-2)=\cdots=\frac{a^2-1}{2}+\frac{a^2+1}{2}=a^2,$$

while that in the second family of pairs being

$$a^2+N=(a^2+1)+(N-1)=\cdots=\frac{a^2+N-1}{2}+\frac{a^2+N+1}{2},$$

forces $a^2+N=b^2$, for some $b=a+2k$. Thus sufficient conditions will be $N=4k(k+a)$, and $1<a<4k$ (so that $a^2<N$).

It is enough to take $a=3k$, for any odd integer $k\ge 1$; then $N=4k(k+a)=(4k)^2$, and $a^2+N=(5k)^2=b^2$. Of course, a more direct approach, based on any other usage of Pythagorean triples, also works. ■

Alternative Solution. (A. Eckstein) We claim all $N=8n$, the multiples of 8, are amiable numbers. Assuming the claim to hold for all $0\le k<n$ for some $n>0$, let us search some $0\le k<n$ such that $(8k+1)+8n$ is a perfect square m^2 (for $n=0$ the claim vacuously holds). Then the set $\{1,2,\dots,8k\}$ can be partitioned in doubletons, each having as sum of its elements a perfect square, since we assume $8k$ is amiable by the induction hypothesis, while the set $\{8k+1,8k+2,\dots,8n\}$ can be partitioned in doubletons

$$\{8k+1,8k+2,\dots,8n\}=\bigcup_{j=1}^{4(n-k)}\{8k+j,8n-j+1\},$$

each having as sum of its elements the perfect square m^2.

For this, it is enough to find an odd m satisfying

$$\sqrt{8n+1}\le m<\sqrt{16n+1},$$

and then take $k=\dfrac{m^2-1}{8}-n$. But for $n=1$ we can take $m=3$, for $n=2$ we can take $m=5$, while for $n\ge 3$ we have $\sqrt{16n+1}-\sqrt{8n+1}\ge 2$, thus we can find such m.

Finally, let us take $n=2^{2p-3}m^p$, when $N=8n=(4m)^p$, which proves infinitely many p-powers are amiable, for any integer $p\ge 2$. ■

Remarks. Two questions naturally come to mind. Do there exist infinitely many even numbers (let alone perfect squares) which are not amiable numbers?[1] Can then one characterize all amiable numbers? The first such amiable number, which is not a multiple of 8, can be checked to be $N=14$ (and $N=18$ is the next)

$$\{1,2,\dots,14\}=\{1,8\}\cup\left(\bigcup_{j=2}^{7}\{j,16-j\}\right).$$

Problem 2. Let ℓ be a line in the plane, and a point $A\notin\ell$. Also let $\alpha\in(0,\pi/2)$ be fixed. Determine the locus of the points Q in the plane, for which there exists a point $P\in\ell$ such that $AQ=PQ$ and $\angle PAQ=\alpha$.

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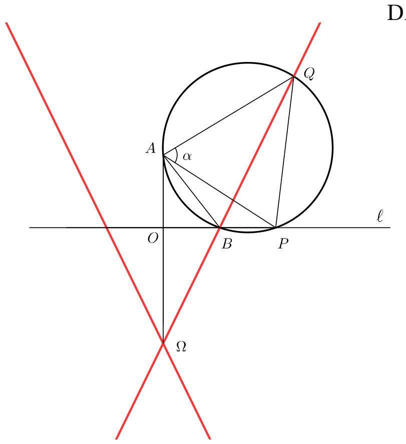

Figure courtesy of ANDREI ECKSTEIN.

Solution. Let the circumcircle of $\triangle APQ$ meet the line ℓ at a second point B. From the concyclicity of points A,B,P,Q follows that the angle made by the lines AB and ℓ is 2α; this means that when $\alpha=\pi/4$ the point B coincides with the foot O of the perpendicular dropped from A onto ℓ, while when $\alpha\ne\pi/4$ the point B must occupy anyone of only two fixed positions on ℓ, B_1 and B_2, symmetrical with respect to O. Of course, there appear two degenerate positions, when $P\equiv B$ and when $Q\equiv B$, but they are trivial.

There exists therefore some point Ω on the line AO (which coincides with O when $\alpha=\pi/4$) so that the locus is made by the $\boxed{\text{two lines through }\Omega\text{, at angles }\pi/2-\alpha\text{ with }AO}$. As for the fullness of the locus, it readily ensues from the fact a construction is possible in all cases (or else by a continuity argument).

The condition $A \notin \ell$ is not strictly necessary; that case is however trivial, in the notations of above, with $A \equiv \Omega$, and the two lines of the locus passing through A and making an angle α with ℓ. For just $\alpha = \pi/4$, the problem has also been **asked to the Juniors**. ■

Analytical Solution. Consider the system of orthogonal coordinates in the plane, such that $A(0, 2\mu)$, the line ℓ is the Ox axis, and $P(2\lambda, 0)$ with λ running over $\mathbb{R}$. Then the midpoint of AP has coordinates $M(\lambda, \mu)$. Let the point $Q(x, y)$, and write the conditions concerning it.

Since $MQ^2 + AM^2 = AQ^2$, we have

$$\left((x-\lambda)^2 + (y-\mu)^2\right) + \left(\lambda^2 + \mu^2\right) = x^2 + (y-2\mu)^2.$$

On the other hand, $MQ = AM \tan PAQ = tAM$ (where we have denoted $t = \tan\alpha$), therefore $MQ^2 = t^2 AM^2$, so

$$(t^2+1)(\lambda^2+\mu^2) = x^2 + (y-2\mu)^2.$$

The equality $AQ^2 = PQ^2$, i.e. $x^2 + (y-2\mu)^2 = (x-2\lambda)^2 + y^2$, yields $\lambda x = (\lambda^2+\mu^2) + \mu(y-2\mu)$. Squaring this up results in $(\lambda^2+\mu^2)x^2 = \left((\lambda^2+\mu^2)+\mu(y-2\mu)\right)^2 + \mu^2 x^2$, which also writes as $(\lambda^2+\mu^2)^2 + 2(\lambda^2+\mu^2)\mu(y-2\mu)) + \mu^2(x^2+(y-2\mu)^2)$, or again $(\lambda^2+\mu^2)^2 + 2(\lambda^2+\mu^2)\mu(y-2\mu)) + \mu^2(t^2+1)(\lambda^2+\mu^2)$. Factoring out $\dfrac{\lambda^2+\mu^2}{t^2+1} \neq 0$, we are left with

$$t^2x^2 - \left((y-2\mu) + (t^2+1)\mu\right)^2 = 0,$$

which writes as $\left(y + (t^2-1)\mu\right)^2 = (tx)^2$, translating into the two lines $y = \pm tx + (1-t^2)\mu$, of slopes $\pm\tan\alpha$, and the same ordinate $(1-t^2)\mu$ at origin (the point Ω of above). ■

Alternative Solutions. (Sketch) Similar computations with the analytic ones above provide a purely trigonometric or vectorial solution. For example, the condition $AQ = PQ$ translates into $\langle \overrightarrow{AM}, \overrightarrow{MQ}\rangle = 0$, writing $\lambda(x-\lambda) - \mu(y-\mu) = 0$, while the condition on the constant angle translates into $\dfrac{\langle \overrightarrow{AQ}, \overrightarrow{AM}\rangle}{AQ \cdot AM} = \cos\alpha$; both of these conditions coming up to identical forms with those obtained in the above. ■

Problem 3. For all triplets a, b, c of (pairwise) distinct real numbers, prove the inequality

$$\left|\frac{a}{b-c}\right| + \left|\frac{b}{c-a}\right| + \left|\frac{c}{a-b}\right| \geq 2$$

and determine all cases of equality.

Prove that if we also impose a, b, c positive, then all equality cases disappear, but the value 2 remains the best constant possible.

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Solution. Denote $x = \dfrac{a}{b-c}$, $y = \dfrac{b}{c-a}$, $z = \dfrac{c}{a-b}$; then it is easily seen that

$$\prod(x-1) = \frac{\prod(a-b+c)}{\prod(b-c)} = \frac{\prod(a+b-c)}{\prod(b-c)} = \prod(x+1),$$

whence $xy+yz+zx = -1$. This relation can also be obtained (rather than guessed) by writing the homogeneous system having a, b, c as unknowns, of determinant

$$\Delta = \begin{vmatrix} 1 & -x & x \\ y & 1 & -y \\ -z & z & 1 \end{vmatrix} = (1+yz) + x(y-yz) + x(yz+z),$$

which computes to $\Delta = 1 + xy + yz + zx$. Since the system originates with a, b, c pairwise distinct, it must also have some other solution than the trivial one $a = b = c = 0$, thus $\Delta = 0$, yielding the seminal relation obeyed by x, y, z.

But since $(xy)(yz)(zx) = (xyz)^2 \geq 0$ we must have at least one of the factors being non-negative, say $xy \geq 0$. Then

$$|x| + |y| + |z| \geq |x+y| + |z| \geq 2\sqrt{|zx+yz|} = 2\sqrt{1+xy} \geq 2.$$

Equality occurs for $xy = 0$, say $x = 0$, and also $|y| = 1$, with $z = -y$, when $a = 0$ and $b = -c$. Thus all equality cases are $\boxed{\{a,b,c\} = \{0, t, -t\}}$, for $t \neq 0$. However, since clearly the value of the expression for a, b, c is the same as that for ta, tb, tc, with $t \neq 0$, the only essential solution is $\boxed{\{0, 1, -1\}}$.

If we also impose $a, b, c > 0$, then the same inequality with respect to 2 holds, but there is no case of equality. The fact we can approach 2 as close as wanted is simply argued by taking an arbitrary $0 < \varepsilon < 1/2$, and $a = \varepsilon^2$, $b = 1+\varepsilon$, $c = 1-\varepsilon$, when

$$\left|\frac{a}{b-c}\right| + \left|\frac{b}{c-a}\right| + \left|\frac{c}{a-b}\right| = \varepsilon\left(\frac{1}{2} + \frac{2+\varepsilon}{1-\varepsilon-\varepsilon^2} - \frac{2-\varepsilon}{1+\varepsilon-\varepsilon^2}\right) + 2,$$

with $\displaystyle\lim_{\varepsilon\to 0} \varepsilon\left(\frac{1}{2} + \frac{2+\varepsilon}{1-\varepsilon-\varepsilon^2} - \frac{2-\varepsilon}{1+\varepsilon-\varepsilon^2}\right) = 0.$ ■

Remarks. Notice for $a+b+c = 0$ the inequality becomes

$$\left|\frac{a+b}{a-b}\right| + \left|\frac{b+c}{b-c}\right| + \left|\frac{c+a}{c-a}\right| \geq 2,$$

a relatively easier inequality, **asked to the Juniors**. Thus, in this case, the two inequalities are equivalent. This can be pushed even further. For a new variable σ, the inequality

$$\left|\frac{a-\sigma}{b-c}\right| + \left|\frac{b-\sigma}{c-a}\right| + \left|\frac{c-\sigma}{a-b}\right| \geq 2$$

can be proved in exactly the same manner, with all equality cases being given by $\{a,b,c\} = \{\sigma, \tau, 2\sigma-\tau\}$, for $\tau \neq \sigma$ (moreover, this allows equality cases even when we impose a, b, c positive, by just taking $0 < \sigma$, $0 < \tau < 2\sigma$). Taking $\sigma = a+b+c$ yields the alternative inequality, this time in all its generality.

Problem 4. The cells of some rectangular $M \times n$ array are colored, each by one of two colors, so that for any two columns the number of pairs of cells situated on a same row and bearing the same color is less than the number of pairs of cells situated on a same row and bearing different colors.

i) Prove that if $M = 2011$ then $n \leq 2012$ (a model for the extremal case $n = 2012$ does indeed exist, but you are not asked to exhibit one).

ii) Prove that if $M = 2011 = n$, each of the colors appears at most $1006 \cdot 2011$ times, and at least $1005 \cdot 2011$ times.

iii) Prove that if however $M = 2012$ then $n \leq 1007$.

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Solution. i) Let's more generally work with $2m-1=2011$. Denote by a_i the number of cells of the first color, and by $b_i=n-a_i$ the number of cells of the second color, situated on row $1\le i\le 2m-1$. Also denote by $A=\sum_{i=1}^{2m-1} a_i$ the total number of cells of the first color, and by N the total number of pairs of cells situated on a same row, but bearing different colors. We proceed by the trusted double counting method.

On one hand, on each row $1\le i\le 2m-1$ we have exactly $a_ib_i=a_i(n-a_i)$ such pairs, so

$$N=\sum_{i=1}^{2m-1} a_i(n-a_i)\le(2m-1)\left(\frac{A}{2m-1}\right)\left(n-\frac{A}{2m-1}\right)$$

by Jensen's inequality, since $f(x)=x(n-x)$ is concave.

On the other hand, the number of such pairs for any two columns is at least m (versus at most $m-1$ pairs of cells situated on a same row and bearing a same color), so $N\ge m\binom{n}{2}=\frac{1}{2}mn(n-1)$.

Putting together the two inequalities from above yields $\frac{A(n(2m-1)-A)}{2m-1}\ge N\ge\frac{1}{2}mn(n-1)$, thus

$$A^2-n(2m-1)A+\frac{1}{2}(2m-1)mn(n-1)\le 0.$$

The discriminant of this trinomial is

$$\Delta=n^2(2m-1)^2-2(2m-1)mn(n-1)=n(2m-1)(2m-n).$$

In order for the inequality to be possible we need $\Delta\ge 0$, thus $n\le 2m$, which in our particular case means $n\le 2012$. It is interesting that we achieved the right bound for n, as it will be seen in the sequel, which will thus provide an alternative proof (based on linear algebra techniques).

ii) The bounds for the number of apparitions of a color are given by the roots of the trinomial at i), which are

$$\frac{\sqrt{n(2m-1)}\left(\sqrt{n(2m-1)}\pm\sqrt{(2m-1)+1-n}\right)}{2}.$$

For $n=2m-1=2011$ this writes $\frac{n(n\pm 1)}{2}$, thus yielding the required bounds. (In fact, they indeed can be reached.)

iii) This is tantamount to doing the computations for $2m$ rather than $2m-1$ rows. Now, in order to have a strictly larger number of pairs of cells of different colors, we need at least $m+1$ of them for any two columns. Putting together the inequalities yields $\frac{A(2mn-A)}{2m}\ge N\ge\frac{1}{2}(m+1)n(n-1)$, thus

$$A^2-2mnA+m(m+1)n(n-1)\le 0.$$

The (reduced) discriminant of this trinomial is

$$\Delta=m^2n^2-m(m+1)n(n-1)=mn(m+1-n).$$

In order for the inequality to be possible we need $\Delta\ge 0$, thus $n\le m+1$, which in our particular case means $n\le 1007$. It is a drastic reduction (to half) of the bound on the number of columns for an odd case $2m+1$, like shown before at i). ■

Remarks. In fact the equality case $n=(2m-1)+1$ leads to $A=m(2m-1)$ (exactly half the elements of the array). This of course means equality in all previous inequalities, thus $a_i=A/(2m-1)=m$ and for any two columns exactly m pairs of cells situated on a same row and bearing different colors. This can in fact be realized, for example for $m=1,2$ or 4 (see models below).

What follows is just presented for educational purpose, as it uses linear algebra techniques in order to provide more insight into the properties of such configurations.

In the sequel the two colors are replaced by the labels $+1$, respectively -1; the n columns become vectors in $\mathbb{R}^{2m-1}$, of pairwise negative dot-products.

$$B_1=\begin{pmatrix}1 & -1\end{pmatrix}_{m=1}\quad B_3=\begin{pmatrix}1 & 1 & -1 & -1\\ 1 & -1 & -1 & 1\\ 1 & -1 & 1 & -1\end{pmatrix}_{m=2}$$

$$B_7=\begin{pmatrix}1 & 1 & 1 & 1 & -1 & -1 & -1 & -1\\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1\\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1\\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1\\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1\\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1\\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1\end{pmatrix}_{m=4}$$

The question is, can such models be found for any other values of m but the powers of 2 ? It can easily be proven that m needs be even (thus no model for $m=3$).

The largest number of pairwise orthogonal vectors in $\mathbb{R}^d$, made of ±1 entries, is clearly not larger than the dimension d of the space. The bound can be shown to be tight for d a power of 2, by using Sylvester's construction for Hadamard matrices (which also extends to provide our above models)

$$H_1=\begin{pmatrix}1\end{pmatrix},\ H_2=\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix},\ \ldots,\ H_{2^{k+1}}=\begin{pmatrix}H_{2^k} & H_{2^k}\\ H_{2^k} & -H_{2^k}\end{pmatrix},\ \ldots$$

We need to also mention Hadamard's conjecture that such models exist whenever $4\mid d$ (it is not difficult to show that $4\mid d$ is a necessary condition, and there are no known counterexamples for it not also being sufficient). We can build a model for a $(d-1)\times d$ matrix made of ±1 entries and pairwise negative dot-products of its column vectors (so $m=d/2$) by taking a Hadamard matrix of order d, and conveniently multiplying its columns by ±1 in order that its last row becomes made of all $+1$'s, then removing it. Thus the question of existence for our models is seen to be equivalent to Hadamard's conjecture. For the particular case $d=2012$, an actual model is now provided by Paley's construction, working when $d-1$ is a power of some prime, since $2011\equiv 3 \pmod 4$ is precisely such a prime! The very Hadamard conjecture is in fact attributed to Paley; see `http://en.wikipedia.org/wiki/Hadamard_matrix`.

Let us now prove that the largest number of vectors in $\mathbb{R}^d$, of pairwise negative dot-products, is $d+1$. The result at point i) of the problem yields $n\le(2m-1)+1$, which indeed will be proven to be the best bound (and this for all vectors, not just those of entries ±1).

Let us proceed by simple induction on d, starting with the obvious case $d = 1$. In dimension $d+1$, let u be one of the vectors. Denote by $U = < u >^{\perp}$ the orthogonal complement of u, of dimension d. Any of the other vectors v can now be uniquely written as $v = v_{\perp} + v_{\dashv}$, with $v_{\dashv} = \dfrac{\langle u, v\rangle}{||u||^2}u$, thus $v_{\perp} \perp u$, i.e. $v_{\perp} \in U$. Notice that the coefficients $\dfrac{\langle u, v\rangle}{||u||^2}$ are negative under our conditions, and also notice we cannot have $v_{\perp} = 0$, since then for a third vector w we would have $0 > \langle v, w\rangle = \langle v_{\dashv}, w\rangle = \dfrac{\langle u, v\rangle \langle u, w\rangle}{||u||^2} > 0$, contradiction.

Let us consider next the orthogonal components $v_{\perp} \in U$, whence $\langle v, w\rangle = \langle v_{\perp} + v_{\dashv}, w_{\perp} + w_{\dashv}\rangle = \langle v_{\perp}, w_{\perp}\rangle + \langle v_{\perp}, w_{\dashv}\rangle + \langle v_{\dashv}, w_{\perp}\rangle + \langle v_{\dashv}, w_{\dashv}\rangle = \langle v_{\perp}, w_{\perp}\rangle + \dfrac{\langle u, v\rangle \langle u, w\rangle}{||u||^2}$, and therefore $\langle v_{\perp}, w_{\perp}\rangle = \langle v, w\rangle - \dfrac{\langle u, v\rangle \langle u, w\rangle}{||u||^2} < 0$, since $\langle v, w\rangle < 0$ and $\langle u, v\rangle \langle u, w\rangle > 0$. We fall under the induction hypothesis, so their number is at most $d+1$, therefore (together with u) there were at most $(d+1)+1$ vectors for dimension $d+1$, and the induction is completed.

In order to build a model, we again reason inductively. For the starting case $d = 1$ we can just take a vector $u \neq 0$ and $-u$. For dimension $d+1$, take an arbitrary vector $u \neq 0$. Take a model made of $d+1$ vectors $v \in U$, the orthogonal complement of u, which is a space of dimension d. Now take a negative real $\lambda < 0$ and consider the vectors $v' = v + \lambda u$. Then $\langle u, v'\rangle = \langle u, v + \lambda u\rangle = \langle u, v\rangle + \lambda \langle u, u\rangle = \lambda ||u||^2 < 0$, and also $\langle v', w'\rangle = \langle v + \lambda u, w + \lambda u\rangle = \langle v, w\rangle + \lambda \langle u, v\rangle + \lambda \langle u, w\rangle + \lambda^2 \langle u, u\rangle = \langle v, w\rangle + \lambda^2 ||u||^2 < 0$ when $\lambda < 0$ is chosen such that $0 < \lambda^2 < \dfrac{1}{||u||^2} \min |\langle v, w\rangle|$ over all the pairs among the $d+1$ vectors of the induction step model. We have thus built with the vectors v' (together with u) a set of $(d+1)+1$ vectors of pairwise negative dot-products in dimension $d+1$.

So that we further our knowledge, and get information on a more relaxed issue, we will also prove that the largest number of not-null vectors in $\mathbb{R}^d$, of pairwise non-positive dot-products, is $2d$.[2]

Let us proceed by simple induction on d, starting with the obvious case $d = 1$. In dimension $d+1$, let u be one of the vectors. Denote by $U = < u >^{\perp}$ the orthogonal complement of u, of dimension d. Any of the other vectors v can now be uniquely written as $v = v_{\perp} + v_{\dashv}$, with $v_{\dashv} = \dfrac{\langle u, v\rangle}{||u||^2}u$, thus $v_{\perp} \perp u$, i.e. $v_{\perp} \in U$. Notice that the coefficients $\dfrac{\langle u, v\rangle}{||u||^2}$ are non-positive under our conditions, and also notice that we can only have $v_{\perp} = 0$ once, for some $u' = v$ (trivially so, since then $u' = u'_{\dashv} = \lambda u$ for some negative real λ).

Let us consider next the orthogonal components $v_{\perp} \in U$, whence $\langle v, w\rangle = \langle v_{\perp} + v_{\dashv}, w_{\perp} + w_{\dashv}\rangle = \langle v_{\perp}, w_{\perp}\rangle + \langle v_{\perp}, w_{\dashv}\rangle + \langle v_{\dashv}, w_{\perp}\rangle + \langle v_{\dashv}, w_{\dashv}\rangle = \langle v_{\perp}, w_{\perp}\rangle + \dfrac{\langle u, v\rangle \langle u, w\rangle}{||u||^2}$, and therefore $\langle v_{\perp}, w_{\perp}\rangle = \langle v, w\rangle - \dfrac{\langle u, v\rangle \langle u, w\rangle}{||u||^2} \leq 0$, since $\langle v, w\rangle \leq 0$ and $\langle u, v\rangle \langle u, w\rangle \geq 0$. We fall under the induction hypothesis, so the number of the not-null vectors is at most $2d$, therefore (together with u and possibly u') there were at most $2(d+1)$ not-null vectors for dimension $d+1$, and the induction is completed.

In order to build a model, notice further that in order to achieve the bound we must have the pair u, u', with all other vectors orthogonal to them. By inductive reasoning it follows that the only possible model is made by some d pairwise orthogonal not-null vectors, and other d obtained from them by multiplication with arbitrary negative scalars.

Of course, the geometric interpretation for negative dot-products is that the vectors make pairwise obtuse angles, while for non-positive dot-products is that the vectors make pairwise obtuse or right angles. Thus we establish reachable bounds for the maximum number of such vectors, in whatever dimension d. When the vectors are made of ± 1 entries, they are among the vertices of the hypercube $\{-1, +1\}^d$.

I apologize for this lengthy development, but I felt these pieces of knowledge are worthwhile to be partaken.

END

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[1] Is it at all curious that the set $\{1, 2, \ldots, N = 2n\}$ can always be decomposed into n pairs, such that the sum of each pair is a prime? This is Greenfield's theorem, and its short and lovely proof makes use of Bertrand's postulate. The link that follows `http://nd.edu/~dgalvin1/pdf/bertrand.pdf` gives quite a worthy reading. The reason to it is that the primes are "more numerous" than squares; their density is $\dfrac{\pi(x)}{x} \sim \dfrac{1}{\ln x}$ $(x \to \infty)$, while that of the squares is $\dfrac{\sqrt{x}}{x} \sim \dfrac{1}{\sqrt{x}}$.

[2] For vectors made of ± 1 entries, the bound can be shown to be tight for $4 \mid d$ whenever a Hadamard matrix H_d of order d does exist (see above), by building a $d \times 2d$ model $A_d = \begin{pmatrix} H_d & -H_d \end{pmatrix}$. The models with ± 1 entries bring now no light on the bounds. For an odd $2m-1$ number of rows the dot-products will have to be negative, so we fall under the conditions of before. For an even $2m$ number of rows the (reduced) discriminant will be $\Delta = m^2 n$ (with bounding roots $m(n \pm \sqrt{n})$), thus establishing no direct restriction on n.