PROBLEMA 1

Soluţie:

Deoarece $\frac{\lg y+\lg z}{2} \geq \sqrt{\lg y \lg z}$ şi analoagele =>

$\sum \lg x \frac{\lg y+\lg z}{2} \geq \sum \lg x \sqrt{\lg y \lg z} \geq$ a ;de unde : $\sum \lg x \lg y \geq$ a

Cum $(\sum \lg z)^2 \geq 3\sum \lg x \lg y \geq 3a$

$\Rightarrow \lg x+\lg y+\lg z \geq \sqrt{3a}$ adica:

$\lg(xyz) \geq \sqrt{3a} \Rightarrow xyz \geq 10^{\sqrt{3a}}$

Egalitate pentru ca $\lg x=\lg y=\lg z \Rightarrow x=y=z$

PROBLEMA 2

Soluţie:

a) Vom arătă că f este strict crescătoare şi prin urmare injectivă. Fie $x_1 > x_2$

Atunci: $f(x_1)-f(x_2)=\frac{a^{x1}+b^{x1}}{c^{x1}+d^{x1}}-\frac{a^{x2}+b^{x2}}{c^{x2}+d^{x2}}=\frac{(a^{x1}+b^{x1})(c^{x2}+d^{x2})-(c^{x1}+d^{x1})(a^{x1}+b^{x1})}{(c^{x1}+d^{x1})(c^{x2}+d^{x2})}=$

$\frac{a^{x1}c^{x2}+a^{x1}d^{x2}+b^{x1}c^{x2}+b^{x1}d^{x2}-c^{x1}a^{x2}-c^{x1}a^{x2}-c^{x1}b^{x2}-d^{x1}a^{x2}-d^{x1}b^{x2}}{(c^{x1}+d^{x1})(c^{x2}+d^{x2})}>0$

Deoarece $a^{x1}c^{x2} > c^{x1}a^{x2} \Leftrightarrow a^{x_1-x_2} > c^{x_1-x_2}$

$a^{x1}d^{x2} > d^{x1}a^{x2} \Leftrightarrow a^{x_1-x_2} > d^{x_1-x_2}$

$b^{x1}c^{x2} > c^{x1}b^{x2} \Leftrightarrow b^{x_1-x_2} > c^{x_1-x_2}$

$b^{x1}d^{x2} > d^{x1}b^{x2} \Leftrightarrow b^{x_1-x_2} > d^{x_1-x_2}$

b) Fie f:R->R, $f(x)=\frac{(\frac{1}{3})^x+(\frac{1}{5})^x}{(\frac{1}{3})^{3x}+(\frac{1}{5})^{3x}}$

Cum $\frac{1}{3}>\frac{1}{5}>\frac{1}{3^3}>\frac{1}{5^3}$ ecuaţia $f(x)=\frac{225}{19}$ (1) are celmult o soluţie(datorită a))

Cum $f(x)=\dfrac{1}{(\frac{1}{9})^x+(\frac{1}{25})^x-(\frac{1}{15})^x}$ (1) se scrie $(\frac{1}{9})^x+(\frac{1}{25})^x-(\frac{1}{15})^x=\frac{19}{225}$

Şi de aici ecuaţia din enunţ are soluţia unică x=-1

PROBLEMA 3

SOLUŢIE:

a) $(1+i)\,(1+2i)\cdot\ldots\cdot(1+2014i) == i^{2014}\cdot\frac{1+i}{i}\cdot\frac{1+2i}{i}\cdot\ldots\cdot\frac{1+2014i}{i} =$

$= i^{4\cdot 503+2}\cdot(1+\frac{1}{i})\,(2+\frac{1}{i})\cdot\ldots\cdot(2014+\frac{1}{i}) == i^2(1-i)\,(2-i)\ldots(2014-i) =$

$= -(1-i)\,(2-i)\ldots(2014-i)$ Rezultă

$(1-i)\,(2-i)\ldots(2014-i) + (1+i)\,(1+2i)\ldots(1+2014i) = 0$

b) Se stie ca radacinile cubice ale unitatii sunt 1, $\varepsilon, \varepsilon^2$ si avem :

$1+\varepsilon+\varepsilon^2=0$, $\bar{\varepsilon}=\frac{1}{\varepsilon}=\varepsilon^2$ $\bar{\varepsilon}^2=\frac{1}{\varepsilon^2}=\varepsilon$

Folosind $|\alpha|^2=\alpha\cdot\bar{\alpha}$, $\forall\alpha\in$ obtinem:

$|z-1|^2+|z-\varepsilon|^2+|z-\varepsilon^2|^2=(\bar{z}-\varepsilon)=(z-1)(\bar{z}-1)+(z-\varepsilon)(\bar{z}-\varepsilon^2)+(z-\varepsilon^2)(\bar{z}-\varepsilon)=$

$=z\bar{z}-z-\bar{z}+1+z\bar{z}-\varepsilon^2 z-\varepsilon\bar{z}+1+z\bar{z}-\varepsilon z-\varepsilon^2\bar{z}+1=$

$=3+3|z|^2-(1+\varepsilon+\varepsilon^2)z-(1+\varepsilon+\varepsilon^2)\bar{z}=3+3|z|^2$

PROBLEMA 4

Solutie:

a) $(b+c)\sin\frac{A}{2}=(b+c)\sqrt{\frac{(p-b)(p-c)}{bc}}\le(b+c)\,\frac{p-b+p-c}{2\sqrt{bc}}$

$=\frac{a(b+c)}{2\sqrt{bc}}\le\frac{a(b+c)2\sqrt{bc}}{4bc}\le\frac{a^2(b+c)^2}{4abc}$ (1)

b) R=$\frac{abc}{4S}$

r=$\frac{S}{P}$

$$\frac{(a+b)(a+c)(b+c)}{16Rr}=\frac{p(a+b)(a+c)(b+c)}{4abc}$$

Însumând (1) trebuie să arătăm că:

$$2a^2(b+c)^2+2b^2(a+c)^2+2c^2(a+b)^2\leq(a+b+c)(a+b)(b+c)(a+c)$$

Rezolvând calculele obţinem:

$$ab(a-b)^2+ac(a-c)^2+bc(b-c)^2\geq 0$$

Egalitatea este adevarata ⇔ a=b=c