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Concursul Interjudeţean „Mathematica – modus vivendi”

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BAREM CLASA a XI-a

1. a. $\det(AB)=\det A\cdot\det B=-1$

$\det A\neq 0, \det B\neq 0\Rightarrow A,B$ inversabile..1p

$(AB)^2-(TrAB)\cdot AB+(\det AB)\cdot I_2=O_2$

$(AB)^2-2014AB-I_2=O_2$ 1p

$A^{-1}ABAB-2014A^{-1}AB-A^{-1}=O_2\Rightarrow BAB-2014B-A^{-1}=O_2\Rightarrow$

$BABB^{-1}-2014BB^{-1}-A^{-1}B^{-1}=O_2\Rightarrow BA-2014I_2-(BA)^{-1}=O_2$ 1

b. Dacă $A=O_2$ se verifică imediat...1p

Pp. $A=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\neq O_2$. Dacă $A^n=O_2\Rightarrow \det(A)=0\Rightarrow A^2=(a+d)\cdot A\Rightarrow$

$\Rightarrow O_2=(a+d)^{n-1}\cdot A$, deci $a+d=0\Rightarrow A^2=O_2$ 1p

$I_2+A+A^2+...+A^{n-1}=I_2+A=\begin{pmatrix} a+1 & b \\ c & d+1\end{pmatrix}$ 1p

$\det(I_2+A+A^2+...+A^{n-1})=(a+1)(d+1)-bc=1$..1p

Total = 7 puncte

2. $(a_n)_{n\geq 1}$ are termeni strict pozitivi şi subunitari, iar $\frac{a_{n+1}}{a_n}=\frac{1}{1+na_n}<1\Rightarrow$

$(a_n)_{n\geq 1}$ convergent...1p

Fie $\lim\limits_{n\to\infty}a_n=l$. Dacă $l>0\Rightarrow \lim\limits_{n\to\infty}(1+na_n)=\infty$ şi $\lim\limits_{n\to\infty}a_{n+1}=\lim\limits_{n\to\infty}\frac{a_n}{1+na_n}=0\Rightarrow l=0$ (F), deci

$\lim\limits_{n\to\infty}a_n=0$ 1p

a. Aplicăm succesiv Lema Stolz-Cesaro...1p

$$\lim_{n\to\infty}\frac{1}{n^4}\left(\frac{b_1}{a_1}+\frac{b_2}{a_2}+...+\frac{b_n}{a_n}\right)=\lim_{n\to\infty}\frac{\frac{b_{n+1}}{a_{n+1}}}{(n+1)^4-n^4}=\lim_{n\to\infty}\frac{b_1+nr}{n}\cdot\frac{\frac{1}{a_{n+1}}}{n^2}\cdot\frac{n^3}{4n^3+6n^2+4n+1}=$$

$$=\frac{r}{4}\lim_{n\to\infty}\frac{\frac{1}{a_{n+1}}}{n^2}=\frac{r}{4}\lim_{n\to\infty}\frac{\frac{1}{a_{n+2}}-\frac{1}{a_{n+1}}}{(n+1)^2-n^2}=\frac{r}{4}\lim_{n\to\infty}\frac{n+1}{2n+1}=\frac{r}{8}$$ 2p

b. Analog $\lim\limits_{n\to\infty}\frac{1}{b_n^4}\left(\frac{b_1}{a_1}+\frac{b_2}{a_2}+...+\frac{b_n}{a_n}\right)=\lim\limits_{n\to\infty}\frac{\frac{b_{n+1}}{a_{n+1}}}{b_{n+1}{}^4-b_n{}^4}=\lim\limits_{n\to\infty}\frac{\frac{b_{n+1}}{a_{n+1}}}{4b_n{}^3+6b_n{}^2r+4b_nr^2+r^3}=$

$=r\cdot\frac{1}{4r^3}\cdot\lim\limits_{n\to\infty}\frac{\frac{1}{a_{n+1}}}{n^2}=\frac{1}{8r^2}$ 2p

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Total = 7 puncte

3. a. $x_{n+1}^2+x_{n+1}\le x_n^2+x_n\Rightarrow y_n=x_n^2+x_n$ este descrescător……………………………1p

Cum y_n este şi mărginit inferior $\Rightarrow$ y_n convergent…………………………….……….1p

De aceea, x_n mărginit (altfel y_n ar avea un subşir nemărginit)……………………..…2p

b. Un exemplu dat justifică cerinţa.

Exemplu: $x_n=\frac{-1+\ -1^{\ n}}{2}$ ………………………………………………………..…3p

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Total = 7 puncte

4. Împărţim matricea X în 4 blocuri 3×3 şi avem:

Dacă X este inversabilă $\Rightarrow X^{-1}=\begin{pmatrix}A' & B'\\ C' & D'\end{pmatrix}$ astfel încât……………………….……..1p

$X\cdot X^{-1}=\begin{pmatrix}AA'+BC' & AB'+BD'\\ CA'+DC' & CB'+DD'\end{pmatrix}=I_2$ ……………………………………………1p

$X=\begin{pmatrix}A & B\\ C & D\end{pmatrix}, A=\begin{pmatrix}1&1&0\\0&1&1\\0&0&1\end{pmatrix}, B=\begin{pmatrix}1&1&0\\0&0&1\\0&0&1\end{pmatrix}, C=\begin{pmatrix}1&0&0\\1&1&0\\0&1&1\end{pmatrix}, D=\begin{pmatrix}1&1&0\\1&1&0\\0&0&1\end{pmatrix}$.................2p

$AA'+BC'=I_3, CA'+DC'=O_3, AB'+BD'=O_3, CB'+DD'=I_3$, de unde rezultă

$B'=-A^{-1}BD', D'=\ D-CA^{-1}B\ ^{-1}, C'=-D'CA^{-1}, A'=-A^{-1}\ I_3+BD'CA^{-1}$ 2p

$D'=\begin{pmatrix}0&0&0\\1&1&-1\\0&0&0\end{pmatrix}, B'=\begin{pmatrix}-1&-1&0\\1&1&-1\\-1&-1&0\end{pmatrix}, C'=\begin{pmatrix}0&0&0\\-1&-1&1\\0&0&0\end{pmatrix}, A'=\begin{pmatrix}-2&-2&1\\2&1&-1\\-2&-1&0\end{pmatrix}\Rightarrow$

$S^2\ X^{-1}\ =25$ 1p

Total = 7 puncte