

32nd International Mathematical Olympiad

First Day — July 17, 1991

Time Limit: $4\frac{1}{2}$ hours

1. Given a triangle ABC , let I be the center of its inscribed circle. The internal bisectors of the angles A, B, C meet the opposite sides in A', B', C' respectively. Prove that

$$\frac{1}{4} < \frac{AI \cdot BI \cdot CI}{AA' \cdot BB' \cdot CC'} \leq \frac{8}{27}.$$

2. Let $n > 6$ be an integer and a_1, a_2, \dots, a_k be all the natural numbers less than n and relatively prime to n . If

$$a_2 - a_1 = a_3 - a_2 = \dots = a_k - a_{k-1} > 0,$$

prove that n must be either a prime number or a power of 2.

3. Let $S = \{1, 2, 3, \dots, 280\}$. Find the smallest integer n such that each n -element subset of S contains five numbers which are pairwise relatively prime.

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1. Suppose G is a connected graph with k edges. Prove that it is possible to label the edges $1, 2, \dots, k$ in such a way that at each vertex which belongs to two or more edges, the greatest common divisor of the integers labeling those edges is equal to 1.

[A *graph* consists of a set of points, called *vertices*, together with a set of *edges* joining certain pairs of distinct vertices. Each pair of vertices u, v belongs to at most one edge. The graph G is *connected* if for each pair of distinct vertices x, y there is some sequence of vertices $x = v_0, v_1, v_2, \dots, v_m = y$ such that each pair v_i, v_{i+1} ($0 \leq i < m$) is joined by an edge of G .]

2. Let ABC be a triangle and P an interior point of ABC . Show that at least one of the angles $\angle PAB, \angle PBC, \angle PCA$ is less than or equal to 30° .

3. An infinite sequence x_0, x_1, x_2, \dots of real numbers is said to be *bounded* if there is a constant C such that $|x_i| \leq C$ for every $i \geq 0$.

Given any real number $a > 1$, construct a bounded infinite sequence x_0, x_1, x_2, \dots such that

$$|x_i - x_j| |i - j|^a \geq 1$$

for every pair of distinct nonnegative integers i, j .