BAREM CLASA a XI-a

Problema1.

Apicand determinantul avem $\det\left(X^2\right)=\left(\det\left(X\right)\right)^2=49\Rightarrow\det\left(X\right)=\pm 7.$

Luam $X=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ din Cayley-Hamilton avem $X^2-tr\left(X\right)X\pm 7I_2=O_2$

Pentru det(X)=7 obtinem solutiile: $A=\begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$ si B=-A.

Pentru det(X)=-7 obtinem solutiile: $C=\frac{i}{\sqrt{3}}\begin{pmatrix} -4 & -5 \\ 5 & 1 \end{pmatrix}$ si D=-C.

Evident $A^{2013}+B^{2013}+C^{2013}+D^{2013}=O_2.$

Problema 2.

(i) Se verifica prin calcul.(3p)

(ii) Ecuatia se scrie $\det\left(A\right)x^2+(tr\left(A\right)tr\left(B\right)-tr\left(AB\right))x+\det\left(B\right)=0.$ Conditia ca sa avem doua solutii distincte (avem ecuatie de gradul doi,det(A)$\neq 0$) este $\left(tr\left(A\right)tr\left(B\right)-tr\left(AB\right)\right)^2\geq 4\det\left(A\right)\det\left(B\right)$ In care folosind punctul (i) si $\det\left(A+B\right)+\det\left(A-B\right)=2\left(\det\left(A\right)+\det\left(B\right)\right)$ obtinem ceea ce se cere demonstrat.(4p)

Problema 3.

Cu inegalitatea mediilor avem :

$$0<\frac{1}{n+1}+\frac{1}{n+2}+...+\frac{1}{n+2^n}\leq\frac{1}{2\sqrt{n}}+\frac{1}{2\sqrt{2n}}+...+\frac{1}{2\sqrt{n2^n}}=\frac{1}{2\sqrt{n}}\left(\frac{1}{1}+\frac{1}{\sqrt{2}}+...+\frac{1}{\sqrt{2^n}}\right)$$

Utilizand Lema Stolz-Cesaro si $\lim_{n\to\infty}\frac{n}{2^n}=0$ avem

$$\lim_{n\to\infty}\frac{\frac{1}{1}+\frac{1}{\sqrt{2}}+...+\frac{1}{\sqrt{2^n}}}{2\sqrt{n}}=\lim_{n\to\infty}\frac{\sqrt{\frac{1}{2^{n+1}}}}{2\left(\sqrt{n+1}-\sqrt{n}\right)}=\lim_{n\to\infty}\frac{1}{2}\sqrt{\frac{1}{2^{n+1}}}\left(\sqrt{n+1}+\sqrt{n}\right)=0.$$

Teorema clestelui ne asigura ca limita ceruta este zero.

PROBLEMA 4.

Deoarece $x_1=\frac{1}{\sqrt{2}}=\cos\frac{\pi}{2^2}$ si folosind formula $\cos\frac{\alpha}{2}=\sqrt{\frac{1+\cos\alpha}{2}}$ obtinem $x_2=\cos\frac{\pi}{2^3}$ si prin inductie matematica $x_n=\cos\frac{\pi}{2^{n+1}}$.(3p)Limita ceruta devine

$$L=\lim_{n\to\infty} x_1x_2...x_n=\lim_{n\to\infty}\left(\cos\frac{\pi}{2^2}\cos\frac{\pi}{2^3}...\cos\frac{\pi}{2^{n+1}}\right).$$

Folosind $\sin 2\alpha=2\sin\alpha\cos\alpha$ obtinem $L=\lim_{n\to\infty}\frac{1}{2^n\sin\frac{\pi}{2^{n+1}}}=\frac{2}{\pi}$.(4p)