CONCURSUL NAŢIONAL DE MATEMATICĂ APLICATĂ
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ETAPA LOCALĂ
Suceava, 22 februarie 2014

Clasa a X-a - profil tehnic, profil servicii şi resurse naturale şi protecţia mediului, profil real-specializarea ştiinţele naturii

BAREM DE CORECTARE

1. Fie $z_n = \left(\frac{1+i}{1-i}\right)^n + \left(\frac{1-i}{1+i}\right)^n, n \in \mathbb{N}$.

a) (3p) Calculaţi z_{2014};

b) (2p) Calculaţi $\overline{z_n}$;

c) (2p) Arătaţi că $z_n \in \mathbb{R}, \forall n \in \mathbb{N}$.

Rezolvare:

a) $z_{2014} = \left(\frac{1+i}{1-i}\right)^{2014} + \left(\frac{1-i}{1+i}\right)^{2014} = \left[\left(\frac{1+i}{1-i}\right)^2\right]^{1007} + \left[\left(\frac{1-i}{1+i}\right)^2\right]^{1007} = -2$

b) $\overline{z_n} = \left(\frac{1-i}{1+i}\right)^n + \left(\frac{1+i}{1-i}\right)^n$

c) $\overline{z_n} = z_n \Rightarrow z_n \in \mathbb{R}$

a) $z_{2014} = \left(\frac{1+i}{1-i}\right)^{2014} + \left(\frac{1-i}{1+i}\right)^{2014}$	1 punct
$z_{2014} = \left[\left(\frac{1+i}{1-i}\right)^2\right]^{1007} + \left[\left(\frac{1-i}{1+i}\right)^2\right]^{1007} = -2$	2 puncte
b) $\overline{z_n} = \left(\frac{1-i}{1+i}\right)^n + \left(\frac{1+i}{1-i}\right)^n$	2 puncte
c) $\overline{z_n} = z_n \Rightarrow z_n \in \mathbb{R}$	2 puncte

2. Fie $S_1 = \frac{1}{\sqrt{2}+1} + \frac{1}{\sqrt{3}+\sqrt{2}} + \frac{1}{\sqrt{3}+\sqrt{2}} + \ldots + \frac{1}{\sqrt{2014}+\sqrt{2013}}$ şi

$$S_2 = \frac{1}{2\sqrt{1}+1\cdot\sqrt{2}} + \frac{1}{3\sqrt{2}+2\sqrt{3}} + \frac{1}{4\sqrt{3}+3\sqrt{4}} + \ldots + \frac{1}{2014\sqrt{2013}+2013\sqrt{2014}}.$$

a) (3p) Calculaţi S_1;

b) (2p) Calculaţi S_2;

c) (2p) Arătaţi că $\left[\frac{S_1}{S_2}\right]=44$.

Rezolvare:

a) $S_1=\frac{1}{\sqrt{2}+1}+\frac{1}{\sqrt{3}+\sqrt{2}}+\frac{1}{\sqrt{3}+\sqrt{2}}+\ldots+\frac{1}{\sqrt{2014}+\sqrt{2013}}=\sqrt{2014}-1$

b) $S_2=1-\frac{1}{\sqrt{2014}}$

c) $\frac{S_1}{S_2}=\sqrt{2014}, \sqrt{2014}\in(44,45)\Rightarrow\left[\frac{S_1}{S_2}\right]=44$

a)Raţionalizarea numitorilor	2 puncte
finalizare	1 punct
b) Raţionalizarea numitorilor	1 punct
$S_2=1-\frac{1}{\sqrt{2014}}$	1 punct
c) $\frac{S_1}{S_2}=\sqrt{2014}$	1 punct
	1punct
$\sqrt{2014}\in(44,45)\Rightarrow\left[\frac{S_1}{S_2}\right]=44$	

3. Fie $a,b,c\in(1,\infty)$.

a) (4p) Arătaţi că $\lg\frac{a+b}{2}\geq\frac{\lg a+\lg b}{2}$;

b) (3p) Arătaţi că $\lg\frac{a+b}{2}\cdot\lg\frac{b+c}{2}\cdot\lg\frac{a+c}{2}\geq\lg a\cdot\lg b\cdot\lg c$.

Rezolvare:

a) $\frac{a+b}{2}\geq\sqrt{ab}\Rightarrow\lg\frac{a+b}{2}\geq\lg\sqrt{ab}\Rightarrow\lg\frac{a+b}{2}\geq\frac{\lg a+\lg b}{2}$

b) Aplicând inegalitatea din punctul a) şi analoagele obţinem prin înmulţire

$\lg\frac{a+b}{2}\cdot\lg\frac{b+c}{2}\cdot\lg\frac{a+c}{2}\geq\frac{\lg a+\lg b}{2}\cdot\frac{\lg b+\lg c}{2}\cdot\frac{\lg a+\lg c}{2}$

$\lg a,\lg b\in(0,\infty)\Rightarrow\frac{\lg a+\lg b}{2}\geq\sqrt{\lg a\lg b}$ de unde obţinem inegalitatea cerută.

a) $a,b\in(1,\infty)\Rightarrow$ deci se poate aplica inegalitatea mediilor	1 punct
$\frac{a+b}{2}\geq\sqrt{ab}\Rightarrow\lg\frac{a+b}{2}\geq\lg\sqrt{ab}$	1 punct
$\lg\frac{a+b}{2}\geq\frac{\lg a+\lg b}{2}$	2 puncte
b) $\lg\frac{a+b}{2}\cdot\lg\frac{b+c}{2}\cdot\lg\frac{a+c}{2}\geq\frac{\lg a+\lg b}{2}\cdot\frac{\lg b+\lg c}{2}\cdot\frac{\lg a+\lg c}{2}$	1 punct
$\lg a,\lg b\in(0,\infty)\Rightarrow\frac{\lg a+\lg b}{2}\geq\sqrt{\lg a\lg b}$ finalizare	1 punct 1punct

4. Fie $z_1,z_2,z_3\in\mathbb{C}^*$ astfel încât $z_1^2+z_2^2+z_3^2=0, z_1+z_2+z_3\neq 0$ şi $|z_1|=|z_2|=|z_3|=1$.

a) (2p) Arătaţi că $(z_1+z_2+z_3)^2=2(z_1z_2+z_3z_2+z_1z_3)$;

b) (2p) Arătaţi că $z_1z_2+z_2z_3+z_1z_3=z_1z_2z_3(\overline{z_1}+\overline{z_2}+\overline{z_3})$;

c) (3p) Arătaţi că $|z_1+z_2+z_3|=2$.

Rezolvare:

a)Folosind identitatea $(z_1+z_2+z_3)^2=z_1^2+z_2^2+z_3^2+2(z_1z_2+z_3z_2+z_1z_3)$ şi $z_1^2+z_2^2+z_3^2=0$

b)Vom avea $z_1z_2+z_2z_3+z_1z_3=z_1z_2z_3\left(\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}\right)$,cum $|z_1|=|z_2|=|z_3|=1\Rightarrow\frac{1}{z_1}=\overline{z_1}$ şi analoagele de unde obţinem cerinţa

c) $(z_1+z_2+z_3)^2=2(z_1z_2+z_3z_2+z_1z_3)\Rightarrow\left|(z_1+z_2+z_3)^2\right|=2|z_1z_2+z_3z_2+z_1z_3|$

$2|z_1z_2+z_3z_2+z_1z_3|=2\left|z_1z_2z_3(\overline{z_1}+\overline{z_2}+\overline{z_3})\right|=2\left|\overline{z_1}+\overline{z_2}+\overline{z_3}\right|=2|z_1+z_2+z_3|$

$\Rightarrow\left|(z_1+z_2+z_3)^2\right|=2|z_1+z_2+z_3|$ de unde obţinem cerinţa

a) $(z_1+z_2+z_3)^2=z_1^2+z_2^2+z_3^2+2(z_1z_2+z_3z_2+z_1z_3)$	1 punct
$z_1^2+z_2^2+z_3^2=0\Rightarrow(z_1+z_2+z_3)^2=2(z_1z_2+z_3z_2+z_1z_3)$	1punct
b) $z_1z_2+z_2z_3+z_1z_3=z_1z_2z_3\left(\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}\right)$	1 punct
$|z_1|=|z_2|=|z_3|=1\Rightarrow\frac{1}{z_1}=\overline{z_1}$ şi analoagele	1punct
c) $\left|(z_1+z_2+z_3)^2\right|=2|z_1z_2+z_3z_2+z_1z_3|$	1punct
$2|z_1z_2+z_3z_2+z_1z_3|=2\left|z_1z_2z_3(\overline{z_1}+\overline{z_2}+\overline{z_3})\right|=2\left|\overline{z_1}+\overline{z_2}+\overline{z_3}\right|=2|z_1+z_2+z_3|$	1punct
finalizare	1punct