Solutii clasa a IX-a

1.Evident $1+kbc\le 1+\frac{k}{2}\left(b^2+c^2\right)=1+\frac{k}{2}\left(1-a^2\right)$ si analoage.Atunci

$$M_s\ge\frac{2a^2}{2+k-ka^2}+\frac{2b^2}{2+k-kb^2}+\frac{2c^2}{2+k-kc^2}=\frac{2}{k}\left(\frac{ka^2}{2+k-ka^2}+\frac{kb^2}{2+k-kb^2}+\frac{kc^2}{2+k-kc^2}\right)=$$

$$=\frac{2}{k}\left(\frac{2+k}{2+k-ka^2}+\frac{2+k}{2+k-kb^2}+\frac{2+k}{2+k-kc^2}-3\right)=\frac{2(2+k)}{k}\left(\frac{1}{2+k-ka^2}+\frac{1}{2+k-ka^2}+\frac{1}{2+k-ka^2}-\frac{3}{2+k}\right)$$

Folosind inegalitatea $\sum\frac{x_i^2}{a_i}\ge\frac{\left(\sum x_i\right)^2}{\sum a_i}$ in

paranteza $\frac{1}{2+k-ka^2}+\frac{1}{2+k-ka^2}+\frac{1}{2+k-ka^2}\ge\frac{9}{6+2k}\left(a^2+b^2+c^2=1\right)$ deci avem

$M_s\ge\frac{2(2+k)}{k}\left(\frac{9}{6+2k}-\frac{3}{2+k}\right)=\frac{3}{3+k}$. Egalitatea se obtine pentru $a=b=c=\frac{1}{\sqrt{3}}$.

2.Fie n=2K sau n=2k+1, $k\ge 1$ .In ambele cazuri multimea de numere intregi $\{0,1,\ldots,k\}$ arata ca $m\ge k+2$. Este natural sa studiem cazul m=k+2.

1)Fie n=2k+1.Fie $x_1,x_2,\ldots,x_{k+2}\in\mathbb{Z}$ si fie $r_1,r_2,\ldots,r_{k+2}\in\{0,1,\ldots,2k\}$ resturile lor la impartirea prin n.Daca exista $i\ne j$ cu $r_i=r_j$,atunci $n|x_i-x_j$ .Daca resturile sunct distincte doua cate doua , atunci cel putin una dintre perechile

$$(1,2k),(2,2k-1),\ldots,(k,k+1)$$

au ambele component in multimea $\{r_1,r_2,\ldots,r_{k+2}\}$ .Resturile egale cu acele componente ne duc la doua numere cu suma divizibila prin n.

2) Cazul n=2k se trateaza analog.Resturile $r_1,r_2,..,r_{k+2}\in\{0,1,\ldots,2k-1\}$ iar rolul perechilor de mai sus este luat de $(1,2k-1),(2,2k-2),\ldots,(k-1,k+1)$.

Asadar in ambele cazuri m=k+2 este cel mai mic numar cu proprietatea ceruta, deci $f(n)=\left[\frac{n}{2}\right]+2.$

Obs.Problema reprezinta o generalizare pentru: Sa se arate ca oricum am alege cinci numere intregi ,exista doua dintre ele care au suma sau diferenta divizibila prin 7,propusa de prof.Ioan Tomescu la Olimpiada judeteana in anul 1981.

3. Fie $f(x)=ax^2+bx+c$ .Avem

$$\sum_{k=1}^{n-1} f(k)=a\frac{n(n-1)(2n-1)}{6}+b\frac{(n-1)n}{2}+(n-1)c=6\frac{n(n-1)}{6}\left((2n-1)\frac{a}{6}+\frac{b}{2}+\frac{c}{n}\right)=0.$$

Deci $\sum_{k=1}^{n-1} f(k)=0$ .Rezulta ca exista $i\neq j$ astfel incat $f(i)f(j)\leq 0$ deci avem o radacina intre i si j.

4.Fie O central cercului circumscris si H_1,H_2,H_3 ortocentrele triunghiurilor ABF,AEC respective DBC.Din formula lui Sylvester avem

$$\overrightarrow{OH_1}=\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OF}$$
$$\overrightarrow{OH_2}=\overrightarrow{OA}+\overrightarrow{OC}+\overrightarrow{OE}$$
$$\overrightarrow{OH_3}=\overrightarrow{OC}+\overrightarrow{OB}+\overrightarrow{OD}$$

De aici obtinem $\overrightarrow{H_2H_1}=\overrightarrow{CF}+\overrightarrow{EB};\overrightarrow{H_2H_3}=\overrightarrow{AD}+\overrightarrow{EB}$ .Din $AD\parallel BE\parallel CF$ rezulta ca exista numerele α si β astfel incat $\overrightarrow{BE}=\alpha\overrightarrow{AD};\overrightarrow{CF}=\beta\overrightarrow{AD}$ de unde obtinem

$\overrightarrow{H_2H_1}=(\beta-\alpha)\overrightarrow{AD};\overrightarrow{H_2H_3}=(1+\alpha)\overrightarrow{AD}$. Ultimile doua egalitati arata ca vectorii $\overrightarrow{H_2H_1},\overrightarrow{H_2H_3},\overrightarrow{AD}$ sunt coliniari,i.e. punctele cerute sunt coliniare pe o dreapta paralela cu dreptele construite.